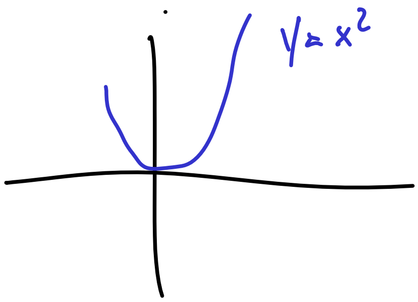
Informally a variety is a collection of points satisfying a system of polynomials equations

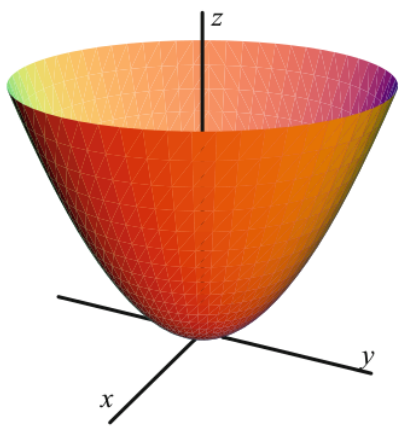
Ex



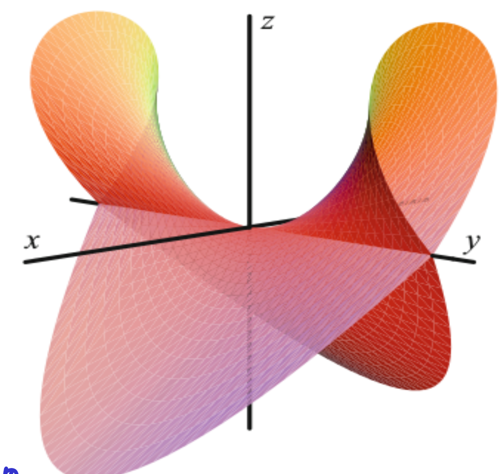
$f = y - x^2$

$V(y - x^2)$ — vanishing of $y - x^2$

$\mathbb{R}[x, y]$



$V(z - x^2 - y^2)$



a field i.e. $k = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$

we work $\underbrace{k[x_1, \ldots, x_n]}$

poly. ring in n- variables

<u>Def</u>

<u>polynomial</u>

$$f = \sum_\alpha \overset{\text{coefficient}}{a_\alpha} x^\alpha, \quad a_\alpha \in k, \quad \alpha = (\alpha_1, \ldots, \alpha_n)$$

$\alpha \in \mathbb{N}^n$

$$x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

↑ monomial

Note

$$x^{(0,\dots,0)} = 1$$

$|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the <u>total degree</u> of x^{α}

max total deg

$$\deg(f) = \max(|\alpha| \mid a_\alpha \neq 0)$$

f divides g

$f \mid g$ if $\exists$ $h \in k[x_1, \dots, x_n]$ s.t.

$$g = fh$$

<u>Def</u>: Affine space (of dim n)

$$k^n = \{ (a_1, \dots, a_n) \mid a_1, \dots, a_n \in k \}$$

polynomials give a map

$$f : k^n \to k$$

$f = 0$ has two meaning, either $f = 0 \in k[x_1, \dots, x_n]$

or $f(a_1, \dots, a_n) = 0$ $\forall$ $(a_1, \dots, a_n) \in k^n$

i.e. f is the zero function.

These are not nessasarily the same (i.e. over a finite field)

$$f = x^2 - x \in \mathbb{Z}/2\mathbb{Z}[x]$$

$f(0) = f(1) = 0$, but $f \neq 0 \in \mathbb{Z}/2\mathbb{Z}[x]$

Prop. Let K be an infinite field, $f \in K[x_1, \dots, x_n]$. Then $f = 0 \in K[x_1, \dots, x_n]$ iff $f : K^n \to K$ is the zero function.

Proof: $f = 0 \in K[x_1, \dots, x_n] \Rightarrow f$ is the zero function.

Now suppose $f(a_1, \dots, a_n) = 0 \quad \forall (a_1, \dots, a_n) \in K^n$, do induction on n

$n=1$ $\quad f(a) = 0 \quad \forall a \in K \Rightarrow f$ has infinitely many roots

$\Rightarrow f = 0 \in K[x_1, \dots, x_n]$

Since a poly of deg m has at most m roots and $\deg(f) < \infty$ (for $f \neq 0$)

Let $f \in K[x_1, \dots, x_n]$, $f(a_1, \dots, a_n) = 0 \quad \forall (a_1, \dots, a_n) \in K^n$

$$f = \sum_{i=0}^{N} g_i(x_1, \dots, x_{n-1}) x_n^i$$

Fix $(a_1, \dots, a_{n-1}) \in K^{n-1}$

$f(a_1, \dots, a_{n-1}, x_n) \in K[x_n]$ and it vanishes at $\forall a_n \in K$

$\Rightarrow$ $f(a_1, \dots, a_{n-1}, x_n) = 0 \in K[x_n]$

$\therefore$ $g_i(a_1, \dots, a_{n-1}) = 0 \quad \forall i$ and $(a_1, \dots, a_{n-1}) \in K^{n-1}$ is arbitrary

$\therefore$ $g_i(a_1, \dots, a_{n-1}) = 0 \quad \forall \; (a_1, \dots, a_{n-1}) \in K^{n-1}$

by induction $g_i = 0 \in K[x_1, \dots, x_{n-1}]$.

$f = 0 \in K[x_1, \dots, x_n]$. ∎

Cor. K infinite. $f, g \in K[x_1, \dots, x_n]$. $f = g$ in $K[x_1, \dots, x_n]$ iff $f : K^n \to K$, $g : K^n \to K$ are the same function.

proof: $f - g$ vanishes everywhere in $K^n \Rightarrow f - g = 0 \in K[x_1, \dots, x_n]$ ∎

Thm Every non constant poly. $f \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

K is <u>algebraically closed</u> if every non-constant poly in $K[x]$ has a root in K, i.e. $\mathbb{R}$ is not alg closed $\mathbb{C}$ is. i.e. $x^2 + 1$.

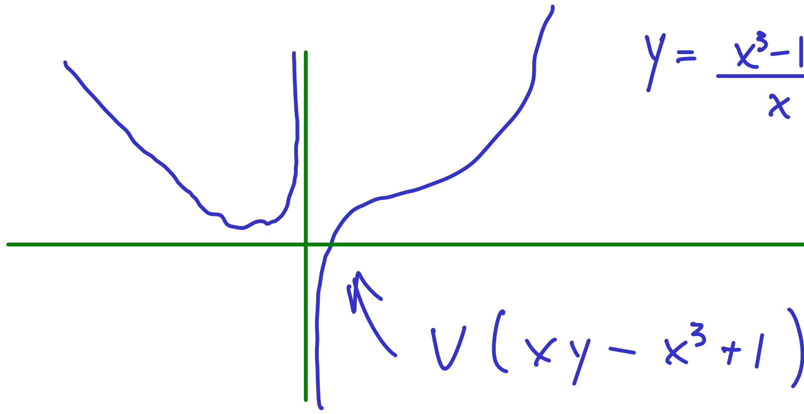
Def: $f_1, \dots, f_s \in K[x_1, \dots, x_n]$. The affine variety defined by $f_1, \dots, f_s$ (or by the (prime) ideal $I = (f_1, \dots, f_s)$) is

$$V(f_1, \dots, f_s) = \{ (a_1, \dots, a_n) \in K^n \mid f_i(a_1, \dots, a_n) = 0 \;\; \forall\, 1 \le i \le s \}$$

i.e. $V(f_1, \dots, f_s)$ is the set of all solutions to

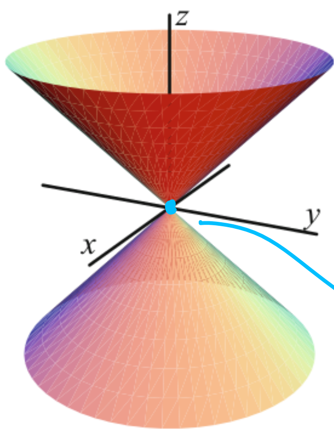
$f_1(x_1, \ldots, x_n) = \cdots = f_s(x_1, \ldots, x_n) = 0.$

More examples



$V(xy - x^3 + 1)$

$y = \frac{x^3 - 1}{x}$ is a variety



$V(z^2 - x^2 - y^2)$

singular point