Problem 1. (15 points.) Let \( f(x) = x^3 + 11x^2 + 13x + 1 \) and let \( \mathbb{Q}[x] \) be the polynomial ring in \( x \) with rational coefficients. Consider the quotient ring \( R = \mathbb{Q}[x]/\langle f(x) \rangle \).

- Is \( R \) a field? Is \( R \) an integral domain? Prove your answers.

Solution: Yes, \( R \) is a field and, hence, also an integral domain.

Proof: We know from a theorem from class (Theorem 16.35 in Judson) that, for \( S \) a ring and \( I \) an ideal in \( S \), \( S/I \) is a field if and only if \( I \) is a maximal ideal. We also know from a theorem from class (Theorem 17.22 of Judson) that if \( F \) is a field, \( p(x) \in F[x] \), then \( \langle p(x) \rangle \) is maximal if and only if \( p(x) \) is irreducible over \( F \). Hence we may conclude that \( R = \mathbb{Q}[x]/\langle f(x) \rangle \) is a field if and only if \( f(x) \) is irreducible over \( \mathbb{Q} \).

Since \( f(x) = x^3 + 11x^2 + 13x + 1 \) has degree three it is reducible if and only if it has a linear factor, that is if and only if \( f(x) \) has a root in \( \mathbb{Q} \). A corollary of Gauss’s Lemma (Corollary 17.15 of Judson) tells us that if \( f(x) \) has a root in \( \mathbb{Q} \) it must have a root \( \alpha \) in \( \mathbb{Z} \), and further, that \( \alpha | 1 \), this would imply that \( \alpha = \pm 1 \). However \( f(1) = 26 \neq 0 \) and \( f(-1) = -2 \neq 0 \), therefore \( f(x) \) is irreducible over \( \mathbb{Q} \) and \( R = \mathbb{Q}[x]/\langle f(x) \rangle \) is a field (and hence an integral domain). ■