Factor Groups = group of cosets of a normal subgroup

**Def:** Let \( N \) be a normal subgroup of \( G \). The factor group or quotient group \( G/N \) is a group consisting of cosets of \( N \) in \( G \) where the operation is given by \((aN)(bN) = abN\), where \( aN, bN \in G/N \) and \((a, b) \in G\).

**Thm:** Let \( G \) be a group, \( N \) a normal subgroup of \( G \). The cosets of \( N \) in \( G \) form a group \( G/N \) of order \([G:N]\) (with op).

**Proof:** First we must show the group op. is well defined (i.e., that different reps. of the same coset yield the same result)

Suppose \( aN = bN \in G/N \) and \( cN = dN \in G/N \). We must show \((aN)(cN) = (bN)(dN)\).

By def \((aN)(cN) = acN\).

Since \( aN = bN \Rightarrow \exists n_1, n_2 \in N \), s.t. \( a = bn_1 \), \( c = dn_2 \),

\((aN)(cN) = acN = b_{n_1}d_{n_2}N, \quad n_2N = N \) since \( n_2 \in N \).
\[ \begin{align*}
&= b n \cdot d N \\
\uparrow \text{ N is a normal subgroup} \\
&= b n \cdot N d \\
\uparrow \text{ N is normal} \\
&= b N d \\
&= b d N = (bN)(dN) \\
\text{: this group operation is well defined} \\
\text{:. } G/N \text{ is a set with a well defined, associative binary operation} \\
\text{identity: } eN = N \\
\text{inverses: } (gN)^{-1} = g^{-1}N \\
\end{align*} \]

\[ [G:N] = \# \text{ of cosets of } N \text{ in } G \]
and \[ G/N \text{ is all cosets of } N \text{ in } G \]
\[ \therefore |a/N| = [G:N] \]

**Ex:** we know one factor group is
\[ \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \]

- Any subgroup \( n\mathbb{Z} \) is normal in \( \mathbb{Z} \) (Since \( \mathbb{Z} \) is Abelian)

\[ \begin{align*}
0 + n\mathbb{Z} &= n\mathbb{Z} \\
1 + n\mathbb{Z} &\text{ } \left\{ \begin{array}{c}
\text{These are the elements of } \mathbb{Z}/n\mathbb{Z} \\
\text{and also the elements of } \mathbb{Z}_n
\end{array}\right\
(n-1) + n\mathbb{Z}
\end{align*} \]
\[ Z_5 = \{ 0, 1, 2, 3, 4 \} \]

\[ \mathbb{Z}/5\mathbb{Z} = \{ \ldots, -5, 0, 5, 10, \ldots \} \]

\[ \text{eq class of } 0 \mod 5 \]

\[ 1 + 5\mathbb{Z} = \{ \ldots, -4, 1, 6, 11, \ldots \} \]

\[ \text{eq. class of } 1 \mod 5 \]

\[ j \equiv k \mod n \iff j - k = \ell n \text{ for some } \ell \in \mathbb{Z} \]

\[ \text{this is precisely the definition of } j - k \in n\mathbb{Z} \]

\[ \text{set of even permutations} \]

\[ N = \{ (1), (132), (123) \} \]

is a normal subgroup of \( S_3 \).

Write mult table for \( S_3/N \):

\[
\begin{array}{c|ccc}
 & N & (12)N \\
\hline
N & N & (12)N \\
(12)N & (12)N & N
\end{array}
\]

Since \( \left[ S_3 : N \right] = 2 \)

\[ S_3/N \text{ is a cyclic group of order } 2 \text{ (generated by } (12)N) \]

\[ S_3/N \cong \mathbb{Z}_2 \]
Ex) \( D_n \) - dihedral group gen by \( r, s \), s.t.

\[
\begin{align*}
    r^n &= 1d \\
s^2 &= 1d \\
r s r s &= r^{-1}
\end{align*}
\]

\( \langle r \rangle = \text{Cyclic subgroup of rotations} \)

\( |\langle r \rangle| = n \)

\( N = \langle r \rangle \) is a normal subgroup

So show \( g N g^{-1} \subseteq N \), show for arbitrary \( g \in D_n \), new

if \( g \in D_n \implies g N \ or \ g = r^k s \ or \ r^k s^{-1} \)

\( \iff g N \ or \ g = r^k s \)

or if \( g = s r^k \), \( n = r^k s \in N \)

\[
\begin{align*}
    s r^k (s r^k)^{-1} &= s r^k r^{-k} s^{-1} \\
    &= s r^k s &= r^{-k} \in N
\end{align*}
\]

\( |D_n/\langle r \rangle| = [D_n: \langle r \rangle] = 2 \)

\( \therefore D_n/\langle r \rangle = \{ \langle r \rangle, s \langle r \rangle \} = \mathbb{Z}_2 \)

Group Homomorphisms

Let \((G, \cdot), (H, \circ)\) be groups

\( \phi: G \rightarrow H \) is a homomorphism if it preserves the group operation i.e.
\[ \phi(g_1g_2) = \phi(g_1) \circ \phi(g_2) \quad \forall \; g_1, g_2 \in G. \]

\[
\begin{align*}
\text{Ex} \quad & G \text{ - group, } \forall \; g \in G \\
\phi : \mathbb{Z} & \rightarrow G \\
\quad n \mapsto g^n & \text{ is a group homomorphism}
\end{align*}
\]

Since \( \phi(m+n) = g^{m+n} = g^m g^n = \phi(m) \phi(n) \)

\( \phi \) maps onto \( <g> \) but Prob. not onto \( G \)

\[
\begin{align*}
\text{Ex} \quad & G = GL_2(\mathbb{R}) = 2 \times 2 \text{ invertible matrices with mul.} \\
\phi : & GL_2(\mathbb{R}) \rightarrow \mathbb{R}^* \\
\quad A \mapsto & \det(A)
\end{align*}
\]

Since \( \phi(AB) = \det(AB) = \det(A) \det(B) = \phi(A) \phi(B) \)

\( \text{and } \det(A) \neq 0 \quad \forall \; A \in GL_2(\mathbb{R}) : \phi(A) \in \mathbb{R}^* \quad \forall \; A \in GL_2(\mathbb{R}) \)

\[
\begin{align*}
\text{Prop} \quad & \text{Let } \phi : G_1 \rightarrow G_2 \text{ be a hom. of groups.} \\
\text{Then:} \\
1) \; & \text{If } e \in G_1 \text{ is the identity then } \phi(e) \text{ is the identity in } G_2 \\
2) \; & \forall \; g \in G_1 \; \phi(g^{-1}) = (\phi(g))^{-1} \\
3) \; & H_1 \text{ subgroup of } G_1 \Rightarrow \phi(H_1) \text{ is a subgroup of } G_2 \\
4) \; & H_2 \text{ is a subgroup of } G_2 \Rightarrow \phi^{-1}(H_2) = \{ g \in G_1 \mid \phi(g) \in H_2 \} \\
\end{align*}
\]
is a subgroup of $G_1$. If $H_2$ is a normal subgroup in $G_2$ then $\Phi^{-1}(H_2)$ is normal in $G_1$.

**Proof:**

1. $\hat{e} = \Phi(e) = \Phi(e) = \Phi(e) = 1$

2. $\Phi(g^{-1})\Phi(g) = \Phi(g^{-1}g) = \Phi(e) = \hat{e}$

   $\therefore \Phi(g^{-1}) = (\Phi(g))^{-1}$

3. Show $\Phi(H_1)$ is a subgroup of $G_2$

   $e e H_1 \implies \Phi(e) = \hat{e} \in \Phi(H_1)$

   $\therefore \Phi(H_1)$ is non-empty and contains $\hat{e} \in G_2$

   Let $x, y \in \Phi(H_1) \implies \exists a, b \in H_1$ s.t. $\Phi(a) = x, \Phi(b) = y$

   Show $x y^{-1} \in \Phi(H_1)$

   $x y^{-1} = \Phi(a)(\Phi(b))^{-1} = \Phi(a b^{-1}) \in \Phi(H_1)$

   $\therefore \Phi(H_1)$ is a subgroup of $G_2$.

4. $H_2$ subgroup of $G_2$, consider the set $H_1 = \Phi^{-1}(H_2)$

   $\hat{e} \in \Phi^{-1}(H_2)$ since $\hat{e} = \Phi(e) \implies \Phi^{-1}(\hat{e}) = \Phi^{-1}(\Phi(e)) = 3 e_3$
If \( a, b \in H_1 \Rightarrow \phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = \frac{e_{H_2} \cdot e_{H_2}}{e_{H_2}} \in H_2 \)

\[ \uparrow \text{by this} \]

For any \( a, b \in H_1 \Rightarrow ab^{-1} \in H_1 = \phi^{-1}(H_2) \)

\[ \therefore H_1 \text{ is a subgroup of } G_1 \]

If \( H_2 \) is normal in \( G_2 \)

Show \( g^{-1}hg \in H_1 \text{ for } h \in H_1 = \phi^{-1}(H_2) \forall g \in G_1 \)

\[ \phi(g^{-1}hg) = \phi(g)^{-1}\phi(h)\phi(g) \]

\[ \in H_2 \text{ since } H_2 \text{ is normal} \]

\[ \Rightarrow g^{-1}hg \in \phi^{-1}(H_2) = H_1 \]

\[ \therefore \phi^{-1}(H_2) \text{ is normal} \]

Def. Let \( \phi: G \rightarrow H \) a group hom. \( e \in H \) the id. in \( H \)

The kernel of \( \phi \) is the subgroup of \( G \)

given by \( \ker(\phi) = \phi^{-1}\{e_H\} = \{g \in G \mid \phi(g) = e_H\} \)

Thm. \( \ker(\phi) \) is a normal subgroup of \( G \).