Example

$D_6$ is the internal direct product

$H = \{ 1, r^3 \}, \quad K = \{ 1, r^2, r^4, s, r^2s, r^4s \}$

- $H \cap K = \{ 1 \}$
- $HK = D_6$
- $hk = kh \quad \forall \ h \in H, \ k \in K$

$D_6 \cong \mathbb{Z}_2 \times S_3$

Example

$S_3$ cannot be written as an internal direct product

$|S_3| = 6 \implies |H| = 3, \ |K| = 2$

The only subgroup of $S_3$ of order 3

$H = \{ 1, (123), (132) \}$

All subgroups of order 2 are $\{ id, 2 \text{-cycle} \}$

but $(2 \text{-cycle})(5 \text{-cycle}) \neq (5 \text{-cycle})(2 \text{-cycle})$ in $S_5$

$(12)(123) = (23)$

Also check

$(123)(12) = (13) \quad \quad (123)(123) \neq (123)(13)$

and $(23)(123) \neq (123)(23)$

\[ \therefore \text{any subgroup of order 2 always has an element which does not commute with } (123) \text{ } \therefore \]

\[ \therefore S_3 \text{ is not an internal direct product.} \]
Thm 1: Let $G$ be the internal direct product of subgroups $H$ and $K$. Then $G = H \times K$.

**Proof:**

$G$ is an internal direct product $\Rightarrow \forall g \in G$

$$g = h_1 k_1, \quad h \in H, \quad k \in K$$

Define $\phi : G \to H \times K$

$$g \mapsto (h, k)$$

Show $\phi$ is well defined I need $h_1 k_1$ to uniquely determine $g$

$g = h_1 k_1 = h'_1 k'_1$

Consider

$$h_1 k_1 = h'_1 k'_1$$

$$h_1 k_1 (k'_1)^{-1} = h'_1 h_1 (k'_1)^{-1}$$

$\in K \quad \in H$

$$k (k'_1)^{-1} = h'_1 h_1 = \epsilon$$

Since $H \cap K = \{ \epsilon \}$

$\therefore$

$$k (k'_1)^{-1} = \epsilon \Rightarrow k = k'_1$$

and $h_1 h'_1 = \epsilon \Rightarrow h = h'_1$

Show $\phi$ preserves group op.

$g_1 = h_1 k_1 \quad g_2 = h_2 k_2$

$$\phi(g_1 g_2) = \phi(h_1 k_1 \cdot h_2 k_2)$$

$h_1$s and $k_1$s commute

$$= \phi(h_1 h_2 k_1 k_2)$$

$$= (h_1, k_1) \cdot (h_2, k_2) = \phi(g_1) \phi(g_2)$$
Ex) \[ \mathbb{Z}_6 = \langle 0, 2, 4 \rangle \times \langle 0, 3 \rangle = \mathbb{Z}_3 \times \mathbb{Z}_2 \]

For a collection of subgroups \( H_1, \ldots, H_n \) of \( G \),
\( G \) is the internal direct product of \( H_1, \ldots, H_n \)
\[ G = H_1 H_2 \cdots H_n = \langle h_1 h_2 \cdots h_n \mid h_i \in H_i \rangle \]
\[ H_i \cap \left( \bigcup_{i \neq j} H_j \right) = \{ e \} \]
\[ h_j h_i = h_i h_j \quad \forall h_i \in H_i \quad \forall h_j \in H_j \]

Thm \[ G \cong H_1 \times H_2 \times \cdots \times H_n \]

Factor Groups and Normal Subgroups

Build a group out of cosets
\[ \mathbb{Z}_5 = \mathbb{Z} / 5\mathbb{Z} \]

Def. A subgroup \( N \) of a group \( G \) is a normal subgroup of \( G \) if \( gN = Ng \quad \forall g \in G \)

Normal subgroup = same right and left cosets for all elements of \( g \).
Every subgroup of an abelian group is normal since all elements commute.

Ex.

\[ H = \left\{ (1), (12) \right\} \text{ in } S_3 \]

\[ \begin{align*}
(123)H &= \left\{ (123), (13) \right\} \quad \text{and} \\
H(123) &= \left\{ (123), (23) \right\}
\end{align*} \]

\[ \therefore H \text{ is not a normal subgroup} \]

\[ N = \left\{ (1), (123), (132) \right\} \text{ is normal} \]

\[ (12)N = N(12) = \left\{ (12), (13), (23) \right\} \]

\[ \therefore N \text{ is a normal subgroup of } S_3. \]

Normal subgroup test

\[ G = \text{group}, \ N \text{ a normal subgroup of } G. \]

The following are equivalent:

1) \( N \) is a normal subgroup in \( G \)

2) \( \forall g \in G, \ gNg^{-1} \subseteq N \)

3) \( \forall g \in G, \ gNg^{-1} = N \)

Proof: \( (1) \Rightarrow (2) \): \( N \) is normal in \( G \) \( \Rightarrow \) \( gN = Ng \ \forall g \in G \)

For a fixed (arbitrary) \( g \in G \) \( \exists n \in N \) s.t.
\[ gn = kg \]

\[ gn^{-1}g^{-1} = k \in N \quad \Rightarrow \quad gN^{-1}g^{-1} \subseteq N \]
2) \Rightarrow 3) \text{ Let } g \in G \text{ be arbitrary.}\)

Suppose \( g Ng^{-1} \subseteq N \), show \( N \subseteq gNg^{-1} \) (i.e., \( N = gNg^{-1} \))

\[
gNg^{-1} \subseteq N \quad \Rightarrow \quad g^{-1} N (g^{-1})^{-1} \subseteq N
g^{-1} ng \in N \quad \forall \ g \in G, \ \forall \ n \in N \quad \Rightarrow \quad g^{-1} ng = \hat{n} \quad \text{for some } \hat{n} \in N \quad \text{and} \quad \forall \ g \in G, \ \forall \ n \in N
n = g \hat{n} g^{-1} \quad \forall \ g \in G
\therefore \ n \in gNg^{-1}
N \subseteq gNg^{-1}
\Rightarrow \ N = gNg^{-1} .

(3) \Rightarrow (1) \text{ Suppose } g Ng^{-1} = N \quad \forall \ g \in G

\Rightarrow \quad \forall \ n \in N \quad \exists \hat{n} \in N \quad \text{s.t.} \quad gng^{-1} = \hat{n}

\Rightarrow \quad gn = \hat{n} g \quad \forall \ g \in G, \ \forall \ n \in N
\therefore \ gN \subseteq Ng

Similarly \quad \forall \ g \in G, \ \forall \ n \in N
Ng \subseteq gN \Rightarrow gN = Ng .
Factor Groups = group of cosets of a normal subgroup

**Def:** Let \( N \) be a normal subgroup of \( G \). The **factor group** or **quotient group** \( G/N \) is a group consisting of cosets of \( N \) in \( G \) where the operation is given by \((aN)(bN) = abN\), where \( aN, bN \in G/N \) and \( ab \in G \).

**Thm:** Let \( G \) be a group, \( N \) a normal subgroup of \( G \). The cosets of \( N \) in \( G \) form a group \( G/N \) of order \([G:N]\) (with operation)

**Proof:** First we must show the group operation is well-defined (i.e., that different reps. of the same coset yield the same result).

Suppose \( aN = bN \in G/N \)

and \( cN = dN \in G/N \)

we must show \((aN)(cN) = (bN)(dN)\)