8. Let p be prime and denote the field of fractions of $\mathbb{Z}_p[x]$ by $\mathbb{Z}_p(x)$. Prove that $\mathbb{Z}_p(x)$ is an infinite field of characteristic p.

We know that $\mathbb{Z}_p(x)$ contains an (isomorphic copy) of $\mathbb{Z}_p[x]$ Since the feild of fractions of an integral domain always contains an isomorphic copy of the integral domain (from a theorem from class). (It is okay if they just say the field contains the poly. ring)

$\mathbb{Z}_p[x]$ is infinite since we may have polynomials of any integer degree, that is for any $n \in \mathbb{Z}$ $\exists$ polynomials $f(x) \in \mathbb{Z}_p[x]$ s.t. $\deg(f) = n$ $\therefore$ $\mathbb{Z}_p[x]$ has more elements than $\mathbb{Z}$ and is, hence, infinite.

Show $\text{char}(\mathbb{Z}_p(x)) = p$: (Either method 1 or 2 is fine)

Method 1:

We know that for any ring with identity the charcterisitic of the ring is the chacterisitic of the identity.

$\therefore$ $\text{char}(\mathbb{Z}_p(x)) = \text{char}(1)$, but $1 \in \mathbb{Z}_p(x)$ is the degree zero poly equal to $1 \in \mathbb{Z}_p$ $\therefore$ $\text{Char}(1) = p$ $\therefore$ $\text{char}(\mathbb{Z}_p(x)) = p$.

Method 2:

$\text{Char}(\mathbb{Z}_p) = p$ and since any element $\gamma \in \mathbb{Z}_p(x)$ has the form

$$\gamma = \frac{a_n x^n + \cdots + a_1 x + a_0}{\underbrace{q(x)}_{\in \mathbb{Z}_p[x]}}, \quad a_n, \ldots, a_1, a_0 \in \mathbb{Z}_p, \; q(x) \in \mathbb{Z}_p[x]$$

since $p \cdot a = 0$ $\forall a \in \mathbb{Z}_p$

Then $$p \cdot \gamma = \frac{(\underbrace{p a_n}_{=0}) x^n + \cdots + (\underbrace{p a_1}_{=0}) x + \underbrace{p a_0}_{=0}}{q(x)} = \frac{0}{q(x)} = 0$$

To see p is the smallest consider the case where some $a_i = 1$, then $m \cdot \gamma \neq 0$ for any $m < p$. $\therefore$ $\text{char}(\mathbb{Z}_p(x)) = p$.