Assignment 6 Solution

27. Let $R$ be a commutative ring. An element $a$ in $R$ is **nilpotent** if $a^n = 0$ for some positive integer $n$. Show that the set of all nilpotent elements forms an ideal in $R$.

**Proof:** Let $I = \{ a \in R \mid a^n = 0 \text{ for some } n \in \mathbb{Z}, n \geq 0 \}$

First show $I$ is an abelian subgroup under addition.

$I$ is non-empty since $0 \in I$. Let $a, b \in I$.

Show $(a - b) \in I$. Since $a, b \in I$ there exist $n_1, n_2$ s.t. $a^{n_1} = b^{n_2} = 0$.

Let $n = \max(n_1, n_2)$, then $a^n = b^n = 0$.

Since $R$ is commutative we have:

$$ (a - b)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} a^i (-b)^{2n-i} $$

Note that every term in the binomial expansion is s.t. either $2n - i \geq n$ or $i \geq n$ and $a^n = b^n = 0$.

$$ \therefore (a - b)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} a^i (-b)^{2n-i} = 0 $$

$$ \therefore a - b \in I $$

Now show $ra \in I$ for all $r \in R$, $a \in I$. Let $a \in I$ s.t. $a^m = 0$ and let $r \in R$ then since $R$ is commutative

$$ (ar)^m = (ra)^m = r^m a^m = r^m \cdot 0 = 0 $$

$$ \therefore ra \in I $$

$$ \therefore I \text{ is an ideal. } $