Check for completion: 9, 10, 18, 25, 27

Marked: #27

Mark Breakdown:
Completion=25 marks, #27=15 marks
Total=40

Assignment 6 Solution

27. Let R be a commutative ring. An element a in R is ***nilpotent*** if $a^n = 0$ for some positive integer n. Show that the set of all nilpotent elements forms an ideal in R.

Proof: Let $I = \{ a \in R \mid a^n = 0 \text{ for some } n \in \mathbb{Z}, n \geq 0 \}$

First show I is an abelian subgroup under addition.

I is non-empty since $0 \in I$. Let $a, b \in I$.

Show $(a-b) \in I$. Since $a, b \in I$ $\exists$ n_1, n_2 s.t. $a^{n_1} = b^{n_2} = 0$.

Let $n = \max(n_1, n_2)$, then $a^n = b^n = 0$.

Since R is commutative we have:

$$(a-b)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} a^i (-b)^{2n-i}$$

Note that every term in the binomial expansion is s.t. either $2n - i \geq n$ or $i \geq n$ and $a^n = b^n = 0$

$$\therefore \quad (a-b)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} a^i (-b)^{2n-i} = 0$$

$\therefore$ $a - b \in I$.

Now show $ra \in I$ for all $r \in R$, $a \in I$. Let $a \in I$ s.t. $a^m = 0$ and let $r \in R$ then

Since R is commutative

$$(ar)^m = (ra)^m = r^m a^m = r^m \cdot 0 = 0$$

$\therefore$ $ra \in I$.

$\therefore$ I is an ideal. $\blacksquare$