§21.4

14. Let K be an algebraic extension of E, and E an algebraic extension of F. Prove that K is algebraic over F. [Caution: Do not assume that the extensions are finite.]

Proof:

We have $F \subset E \subset F$. Let $\alpha \in K$, show that α is algebraic over F. By assumption K is algebraic over E, hence any $\alpha \in K$ is the root of some polynomial in $E[x]$, and in particular, there exists a minimal polynomial $p(x) \in E[x]$ such that $p(\alpha) = 0$. Write this polynomial as

$$p(x) = x^n + \beta_{n-1}x^{n-1} + \cdots + \beta_1 x + \beta_0 \in E[x].$$

Consider the field extension $L = F(\beta_{n-1}, \ldots, \beta_0)$ of F, since each $\beta_j \in E$ and since E is algebraic over F then, by a theorem from class (Theorem 21.22 of Judson), L is a finite extension of F. Hence $[L : F] < \infty$. Now consider the extension $L(\alpha)$ of L, we know α is algebraic over L since α is the root of the degree n polynomial $p(x)$, and $p(x) \in L[x]$ because all its coefficients are in L by construction. Since $L(\alpha)$ is a simple extension and α is algebraic over L then, by a theorem from class (Theorem 21.13 of Judson), $[L(\alpha) : L] = n$, and in particular, $[L(\alpha) : L]$ is finite.

Note that we have $F \subset L \subset L(\alpha)$, so by a theorem from class (Theorem 21.17 of Judson) we have that

$$[L(\alpha) : F] = [L(\alpha) : L][L : F].$$

By the arguments above $[L(\alpha) : L]$ and $[L : F]$ are both finite, hence $[L(\alpha) : F]$ is finite. Thus we have that $F(\beta_{n-1}, \ldots, \beta_0, \alpha)$ is a finite extension of F, and hence by a theorem from class (Theorem 21.22 or Theorem 21.15 of Judson) we have that $F(\beta_{n-1}, \ldots, \beta_0, \alpha)$ is an algebraic extension of F, and in particular that α is algebraic over F. Therefore all $\alpha \in K$ are algebraic over F, which is what we wished to show. $\square$