Def: A permutation is even if it can be expressed as an even number of transpositions.

The Alternating Group: \( (\text{on } n \text{ letters}) \)

\[ A_n = \text{set of all even permutations in } S_n \]

Theorem 5.16 \( \) The set \( A_n \) is a subgroup of \( S_n \).

Proof:
- Product of two even permutations is even: \( A_n \) is closed.
- \( \text{id} \) is even (Theorem from Friday): \( \text{id} \in A_n \).
- If \( \sigma \) is even, \( \sigma = \sigma_1 \cdots \sigma_r \) for \( r \) even.

\[ (\sigma_1 \cdots \sigma_r)^{-1} = \sigma_r^{-1} \cdots \sigma_1^{-1} = \sigma_r \cdots \sigma_1 \]

\[ \therefore \sigma^{-1} \in A_n. \]

Proposition 5.17 \( \) For \( n \geq 2 \) the number of even permutations is equal to the number of odd permutations \( \Rightarrow |A_n| = \frac{n!}{2} \).

Proof:
- \( A_n \) - even perm
- \( B_n \) - odd perm

Show \( \exists \) a bijection between \( A_n \) and \( B_n \).

Fix arbitrary transposition \( \sigma \in S_n \) (\( \exists \) since \( n \geq 2 \)).

Define a map \( \lambda_\sigma: A_n \rightarrow B_n \)

\[ : T \mapsto \sigma \cdot T \]
1-1: Suppose $\lambda_0(T) = \lambda_0(M)$ for $T, M \in \text{An}$ then

$$\sigma T = \sigma M$$

$$T = \sigma^{-1} \sigma T = \sigma^{-1} \sigma M = M$$

$$\therefore T = M.$$

$$\therefore \lambda_0 \text{ is 1-1.}$$

Onto: Pick arbitrary $\lambda \in B_n$ show $\exists T \in \text{An}$ s.t $\lambda_0(T) = \lambda$

Consider $T = \sigma \lambda$, since $\lambda$ is odd $\therefore T$ is even and

$$\lambda_0(T) = \sigma \sigma \lambda = \lambda$$

Example 5.18. The group $A_4$ is the subgroup of $S_4$ consisting of even permutations. There are twelve elements in $A_4$:

- $(1)$
- $(12)(34)$
- $(13)(24)$
- $(14)(23)$
- $(123)$
- $(132)$
- $(124)$
- $(142)$
- $(134)$
- $(143)$
- $(234)$
- $(243)$

**Dihedral Groups**

**Subgroups of $S_n$**

$n^{th}$ dihedral group: group of rigid motions of a regular $n$-gon

- Notice we have $n$ choices for the first vertex
- If we replace 1 by $k$ then 2 must be one of $k+1$ or $k-1$
- $2n$ possible rigid motions
  - $n$ reflections and $n$ rotations

**Figure 5.19:** A regular $n$-gon
Theorem 5.20

The dihedral group $D_n$, is a subgroup of $S_n$ of order $2n$.

![Diagram of rotations and reflections of a regular $n$-gon](image1)

**Figure 5.21**: Rotations and reflections of a regular $n$-gon

![Diagram of types of reflections of a regular $n$-gon](image2)

**Figure 5.22**: Types of reflections of a regular $n$-gon

Theorem 5.23

The group $D_n$ in $n \geq 3$ consists of all products of two elements $r$ and $s$ satisfying the relations

- $r^n = 1$ (rotations)
- $s^2 = 1$ (reflections)
- $srs = r^{-1}$

**Proof**: The are exactly $n$-rotations:

- $id$, $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\ldots$, $\frac{(n-1)2\pi}{n}$.
\[ r = \frac{2\pi}{n} \quad \text{this generates all other rotations} \]
\[ \text{(think of roots of unity)} \]
\[ \text{i.e.} \quad r^k = k \cdot \frac{2\pi}{n} \]

Label \( n \) reflections \( S_1, \ldots, S_n \) where \( S_k \) leaves the \( k \)th vertex fixed. Two cases

Even \( n \) vertices
- Two vertices fixed by such a reflection

Odd \( n \) vertices
- One vertex fixed
\[ |S_n| = 2 \]

\( S = S_1 \) Then \( S^2 = \text{id} \) \( r^n = \text{id} \)

Consider the first vertex of an \( n \)-gon:
Any rigid motion replace \( 1 \) by \( k \)
then \( 2 \) becomes either \( k+1 \) or \( k-1 \)

If \( 2 \Rightarrow k+1 \) then
\[ t = r^{k-1} \]

If \( 2 \) is replaced by \( k-1 \) then
\[ t = r^{k-1} \]

Show \( r^{-1} = S r S \)

\[ S r S = \text{First} \]

Show \( r^{-1} = S r S \)
Example: $D_4$ rigid motions of a square

$|D_4| = 8$

Figure 5.25: The group $D_4$

Rotations

$r = (1 2 3 4)$
$r^2 = (1 3)(2 4)$
$r^3 = (1 4 3 2)$
$r^4 = (1)$

Reflections

$s_1 = (2 4)$
$s_2 = (1 3)$

The other two reflections

$rs_1 = (1 2)(3 4)$
$r^3 s_1 = (1 4)(2 3)$ — Reflection in "y" axis's