• \( N \) is a normal subgroup if \( gH = Hg \ \forall \ g \in G \).

• If \( N \) is a normal subgroup of \( G \), we can define

\[
G/N = \left\{ aN \mid a \in G \right\}
\]

Operation on \( G/N \)

\[
(aN) \cdot (bN) = (abN)
\]

factor group or quotient group

\[
\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n
\]

\[
\downarrow \text{coset} \quad 0+n\mathbb{Z}, \ 1+n\mathbb{Z}, \ 2+n\mathbb{Z}, \ldots \ n-1+n\mathbb{Z}
\]

**Example**

\[
D_4 / R_4 \cong \mathbb{Z}_2
\]

\[
\downarrow
\]

\[
L \langle r \rangle = \langle r \rangle, \ \langle s \rangle, \ \langle rs \rangle
\]

\[
n^4 = 1
\]

\[
s^2 = 1
\]

\[
rs = sr^{-1}
\]

• Homomorphism

\[
\phi : G \rightarrow H
\]

\[
\phi( g_1 \cdot g_2 ) = \phi(g_1) \cdot \phi(g_2)
\]

\[
\phi(e_G) = e_H, \ \phi(g^{-1}) = \phi(g)^{-1}, \ \phi(\langle a \rangle) \text{ is a subgroup...}
\]
\[ \ker(\phi) = \{ g \in G \mid \phi(g) = e_H \} \]

**Normal subgroup of** \( G \).

*Canonical homomorphisms (\( H \) a normal subgroup)*

\[ \phi : G \rightarrow G/H \]

\[ g \mapsto gH \]

Note \( \ker(\phi) = H \)

**First Isomorphism Theorem**

Let \( \psi : G \rightarrow H \) be a homomorphism, \( K = \ker(\psi) \) (is a normal subgroup of \( G \)). Let \( \phi : G \rightarrow G/\ker(\psi) \) be the canonical homomorphism. \( \exists \) unique isomorphism \( \pi \)

\[ \pi : G/\ker(\psi) \rightarrow \psi(G) \]

\[ \text{s.t. } \psi = \pi \circ \phi \]

\[ \begin{array}{ccc}
G & \xrightarrow{\psi} & H \\
\phi \downarrow & & \downarrow \pi \\
G/\ker(\psi) & \xrightarrow{\pi} & \psi(G) \\
\end{array} \]

\[ \therefore \psi(g) = \pi \circ \phi(g) \]

**Proof:** \( K = \ker(\psi) \) is normal in \( G \).

Define \( \pi : G/K \rightarrow \psi(G) \)

\[ gK \mapsto \psi(g) \]
Show that \( \eta \) is well defined. If \( g_1 k = g_2 k \) then we have \( k_1, k_2 \in \ker(\eta) \):

\[
\begin{align*}
g_1 k_1 &= g_2 k_2 \\
g_1 k_1 k_2^{-1} &= g_2
\end{align*}
\]

\( k = k_2^{-1} k_1 \in \ker(\eta) \).

\[
\eta(g_1, k) = \psi(g_1) \psi(k) = \psi(g_1 k) = \psi(g_2) = \eta(g_2, k)
\]

\( \therefore \eta \) is well defined.

Show \( \eta \) is a homomorphism

\[
\eta(g_1, k, g_2, k) = \eta(g_1 g_2, k) = \psi(g_1 g_2)
\]

\[
= \psi(g_1) \psi(g_2)
\]

\[
= \eta(g_1, k) \eta(g_2, k)
\]

\( \eta \) is onto by def.

Show 1-1. Say \( \eta(g_1, k) = \eta(g_2, k) \)

\[
\therefore \psi(g_1) = \psi(g_2)
\]

\[
e_H = (\psi(g_1))^{-1} \psi(g_2)
\]

\[
= \psi(g_1^{-1}) \psi(g_2)
\]

\[
= \psi(g_1^{-1} g_2)
\]

\[
\Rightarrow e_H = g_1^{-1} g_2 \Rightarrow g_1 = g_2
\]

\( \therefore \) 1-1.
Second isomorphism theorem

$H$ subgroup of $G$, $N$ normal subgroup of $G$

Then

- $HN$ a subgroup of $G$
- $HN$ a normal subgroup of $H$
- $H/(H \cap N) \cong HN/N$

Correspondence theorem

$H$ a subgroup, $N$ a normal subgroup of $G$

$H \rightarrow H/N$

$\{ \text{subgroups of } H \text{ containing } N \} \rightarrow \{ \text{subgroups of } G/N \}$

Third isomorphism theorem

$G$ a group, $H, N$ normal subgroups

$G/H \cong \frac{G/N}{H/N}$

Example

$\mathbb{Z}/m \mathbb{Z} \cong \left( \mathbb{Z}/m \mathbb{Z} \right)/\left( m \mathbb{Z}/m \mathbb{Z} \right)$