Cosets:

Let \( G \) be a group, \( H \) a subgroup, \( g \in G \).

Define a left coset of \( H \) with representative \( g \) as
\[
gH = \{ gh \mid h \in H \}
\]

Right coset
\[
Hg = \{ hg \mid h \in H \}
\]

Ex.

Let \( H = \langle 3 \rangle = \{0, 3, 6\} \) be the subgroup of \( \mathbb{Z}_6 \) generated by 3.

Cosets are:
\[
0 + H = 3 + H = \{0, 3\}
\]
\[
1 + H = 4 + H = \{1, 4\}
\]
\[
2 + H = 5 + H = \{2, 5\}
\]

Ex.

Let \( k \) be the subgroup of \( S_3 \) given by \( k = \langle (1), (12) \rangle \).

Left cosets
\[
(1)k = (12)k = \{ (1), (12) \}
\]
\[
(13)k = (123)k = \{ (13), (123) \}
\]
\[
(23)k = (132)k = \{ (23), (132) \}
\]

Right cosets
\[
k(1) = k(12) = \{ (1), (12) \}
\]
\[
k(13) = k(132) = \{ (13), (132) \}
\]
\[
k(23) = k(123) = \{ (23), (123) \}
\]
Lemma 6.3. Let $H$ be a subgroup of a group $G$ and suppose that $g_1, g_2 \in G$. The following conditions are equivalent.

1. $g_1 H = g_2 H$;
2. $H g_1^{-1} = H g_2^{-1}$;
3. $g_1 H \subseteq g_2 H$;
4. $g_2 \in g_1 H$;
5. $g_1^{-1} g_2 \in H$.

Theorem 6.4

Let $H$ be a subgroup of a group $G$. Then the left (resp. right) cosets of $H$ in $G$ partition $G$.

That is, $G$ is the disjoint union of the left (resp. right) cosets of $H$ in $G$.

Proof:

Let $g_1 H$, $g_2 H$ be two cosets of $H$ in $G$. Show that

$g_1 H \cap g_2 H = \emptyset$ or $g_1 H = g_2 H$.

Suppose $g_1 H \cap g_2 H \neq \emptyset$ and $a \in g_1 H \cap g_2 H$ then

$a = g_1 h_1 = g_2 h_2$ for some $h_1, h_2 \in H$

$\Rightarrow g_1 = g_2 h_2 h_1^{-1}$

$g_1 = g_2 H$

$g_1 H = \frac{1}{2} g_1 H \mid h e H \Rightarrow \frac{1}{2} g_2 H \mid h e H \Rightarrow$

$\frac{1}{2} g_2 H \mid h e H \Rightarrow g_2 H$

And all $g \in G$ appear in some coset, in particular in $g H$ since $g e = g$ and $e \in H$.\[\]
Def | Let $G$ be a group, $H$ is a subgroup. Define the index of $H$ in $G$:

$$[G : H] = \text{no. of left cosets of } H \text{ in } G$$

Ex | $G = \mathbb{Z}_6$, $H = \{0, 3\}$, $[G : H] = 3$

Theorem 6.8 | Let $H$ be a subgroup of $G$.

If $H$ left cosets of $H$ in $G$, then $H$ right cosets of $H$ in $G$.

Proof:

$L_H$ - left cosets
$R_H$ - right cosets

We wish to define a bijection $\phi : L_H \rightarrow R_H$.

If $gH \in L_H$, let $\phi(gH) = Hg^{-1}$, note that this map is well-defined since if $g_1H = g_2H = gH$ are different representatives of $gH$ then $Hg_1^{-1} = Hg_2^{-1} = Hg^{-1}$ (by Lemma 6.3 Part 1 and 2).

1-1: Suppose $\phi(g_1H) = \phi(g_2H)$

$\Rightarrow Hg_1^{-1} = Hg_2^{-1} \Rightarrow g_1H = g_2H$.

Onto: For any $Hg \in R_H$, we have that $\phi(g^{-1}H) = H(g^{-1})^{-1} = Hg$. Thus $\phi$ is onto.

$\Rightarrow \phi$ is a bijection $|L_H| = |R_H|$. 
Lagrange's Theorem

**Theorem**

Let \( G \) be a finite group and let \( H \) be a subgroup of \( G \).

Then \[ \frac{|G|}{|H|} = [G : H] = \text{is the number of distinct left cosets of } H \text{ in } G. \]

In particular \[ |H| \mid |G|. \]

**Proof:** (by Theorem 6.4)

The group \( G \) is partitioned into \([G : H]\) distinct cosets.

Each coset has \(|H|\) elements.

\[\therefore \quad |G| = [G : H]|H| \]

**Corollary**

Suppose \( G \) is a finite group, \( g \in G \).

Then \(|g| \mid |G|\). That is order of an element divides order of \( G \).

**Proof:** Apply Lagrange's Theorem to \( H = \langle g \rangle \).

**Corollary**

Let \(|G| = p\) for \( p \) prime. Then \( G \) is cyclic and is generated by any \( g \in G \) s.t. \( g \neq e \).

**Proof:**

Let \( g \in G, g \neq e \).

Then \(|g| \mid |G|\), but since \( g \neq e \) \(|g| > 1\) and \(|g| \leq |G|\).

\[\Rightarrow \quad |g| = |G| \Rightarrow G = \langle g \rangle.\]
Corollary 1 Let $H$ and $K$ be subgroups of $G$, $|G| < \infty$ such that $K \leq H \leq G$. Then $[G:K] = [G:H][H:K]$.

Proof. From Lagrange's Theorem

$$[G:K] = \frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = [G:H][H:K].$$