Theorem Let F be a field and let $p(x)$ be a non-constant polynomial in $F[x]$. Then there exists an extension field E of F and an elment $\alpha \in E$ s.t. $p(\alpha) = 0$.

Proof: we may assume $p(x)$ is irreducible in $F[x]$ (since we could factor into irreducibles)

- want to find an extension feild E of F s.t $p(\alpha)=0$ for some $\alpha \in E$.

Consider
$$E = F[x] / \langle p(x) \rangle$$

$\because$ $p(x)$ is irreducible $\Rightarrow$ $\langle p(x) \rangle$ is maximal

$\therefore$ E is a feild.

- First show E is an exension feild

[Since $\deg(p(x)) \geq 1$ we would expect F would be rep. by $a + \langle p(x) \rangle$ $a \in F$]

Define a hom. $\psi : F \to F[x]/\langle p(x) \rangle$
$$a \longmapsto a + \langle p(x) \rangle$$

$$\psi(a) + \psi(b) = (a + \langle p(x) \rangle) + (b + \langle p(x) \rangle)$$
$$= (a+b) + \langle p(x) \rangle = \psi(a+b)$$

$$\psi(a)\psi(b) = (a + \langle p(x)\rangle)(b + \langle p(x)\rangle)$$
$$= ab + \langle p(x)\rangle = \psi(ab).$$

Show ψ 1-1:

if $a + \langle p(x)\rangle = \psi(a) = \psi(b) = b + \langle p(x)\rangle$

$\Rightarrow$ $a - b + \langle p(x)\rangle = \langle p(x)\rangle$

$\Rightarrow$ $a - b \in \langle p(x)\rangle$, but $\deg(p(x)) \geq 1$

$\therefore$ $a - b = 0$ $\Rightarrow$ $a = b$ $\therefore$ ψ is 1-1.

$\therefore$ $\ker\psi = 0$

$\therefore$ By the first iso. Theorem

$$F \cong \underset{\uparrow \text{ sub feild of } E}{\psi(F)} = \{a + \langle p(x)\rangle \mid a \in F\}$$

$\therefore$ E is an exension feild of F.

Now show $\exists$ $\alpha \in E$ s.t. $p(\alpha) = 0$

Take $\alpha = x + \langle p(x)\rangle \in E$

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$.

Evaluate at α

$$P(\alpha) = a_0 + a_1(x + \langle p(x)\rangle) + \cdots + a_n(x + \langle p(x)\rangle)^n$$

$$= a_0 + a_1(x + \langle p(x)\rangle) + \cdots + a_n(x^n + \langle p(x)\rangle)$$

$$= \underbrace{(a_0 + a_1 x + \cdots + a_n x^n)}^{=p(x)} + (\underbrace{a_1\langle p(x)\rangle + \cdots + a_n\langle p(x)\rangle}_{=\langle p(x)\rangle})$$

$$= p(x) + \langle p(x)\rangle$$

$$= 0 + \langle p(x)\rangle$$

$\therefore$ $p(\alpha) = 0$ in E.

$\therefore$ α is a zero of $p(x)$ in $E = F[x]/\langle p(x)\rangle$.

$\blacksquare$.

<u>Ex)</u> Let $p(x) = x^5 + x^4 + 1 \in \mathbb{Z}_2[x]$

$$= (x^2+x+1)(x^3+x+1)$$

to find an extension feild E s.t. $p(x)$ has a root in E, take

$$E = \mathbb{Z}_2[x]/\langle x^2+x+1\rangle$$

or

$$E = \mathbb{Z}_2[x]/\langle x^3+x+1\rangle.$$

Algebraic Elements

- $\alpha \in E$, E an extension feild over F is <u>algebraic</u> over F if $f(\alpha) = 0$ for some $f(x) \neq 0 \in F[x]$.
- An element β in E is <u>not</u> algebaic, or is <u>trancentental</u> over F if it is not algebraic, i.e. $\exists$ no $f(x)$ s.t. $f(\beta) = 0$.
- E is an <u>algebraic extension</u> of F if every $\alpha \in E$ is algebraic over F.
- If E is a field extension of F, $\alpha_1, \ldots, \alpha_n \in E$
  $F(\alpha_1, \ldots, \alpha_n) =$ smallest field containing $\alpha_1, \ldots, \alpha_n$.
- $E = F(\alpha)$ for some $\alpha \in E$
  ↑ E is a <u>simple extension</u> of F.

Ex) $\sqrt{2} \Leftrightarrow$ zero of $x^2 - 2$

$i \Leftrightarrow$ zero of $x^2 + 1$

$\therefore$ $\sqrt{2}$, i are algebraic over $\mathbb{Q}$.

π, e are algebraic over $\mathbb{R}$, but <u>trancendetal</u> over $\mathbb{Q}$

Almost all real numbers are trancedental over $\mathbb{Q}$.

Def A complex number that is algebraic over $\mathbb{Q}$ is an <u>algebraic number</u>.

otherwise a trancedental number.

Ex $\alpha = \sqrt{2+\sqrt{3}}$ is algebraic over $\mathbb{Q}$.

$$\alpha^2 = 2+\sqrt{3} \Rightarrow \alpha^2 - 2 = \sqrt{3}$$

$$\Rightarrow (\alpha^2-2)^2 = 3$$

$$\Rightarrow \alpha^4 - 4\alpha^2 + 1 = 0$$

$\therefore$ α is a root of $f(x) = x^4 - 4x^2 + 1 \in \mathbb{Q}[x]$

Theorem Let E be an extension field of F, $\alpha \in E$

α is trancendental over F if and only if

$F(\alpha) \simeq F(x)$

$\uparrow$ the feild of fractions of $F[x]$.

Proof: