Prop. Let \( S = \{v_1, \ldots, v_n\} \) be vectors in a vector space \( V \). 
\( \text{Span}_F(S) \) is a subspace of \( V \).

Linear Independence

Def. A set of vectors \( v_1, \ldots, v_n \) is linearly independent if

\[ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \]

implies that \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \).

Linearly dependent if there are non-zero \( \alpha_i \)'s s.t.

\[ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \]

Prop. Let \( \{v_1, \ldots, v_n\} \) be a linearly independent set in a vector space \( V \).

Suppose 
\[ \alpha_1 v_1 + \cdots + \alpha_n v_n = 0 \]

\[ \Rightarrow \alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n \]

Proof:

\[ \alpha_1 v_1 + \cdots + \alpha_n v_n = 0 \]

\[ \Rightarrow (\alpha_1 - \beta_1)v_1 + \cdots + (\alpha_n - \beta_n)v_n = 0 \]

Since \( \{v_1, \ldots, v_n\} \) are lin. independent.
Prop 1: \( \exists u_1, \ldots, u_n \) are linearly dependent \( \iff \) one of the \( u_i \)'s is a linear combo. of the rest.

Prop 1: Suppose \( V = \text{Span}_F (u_1, \ldots, u_n) \) with \( u_1, \ldots, u_n \) lin. independent.

If \( m > n \) then any set of \( m \) vectors in \( V \) must be lin. dependent.

Def 1: \( \{ e_1, \ldots, e_n \} \) is a basis of \( V \) if \( \{ e_1, \ldots, e_n \} \) is linearly independent and

\[ V = \text{Span}_F (e_1, \ldots, e_n) \]

Ex 1: \((1,1,0), \ldots \) etc. \( \mathbb{R}^3 \)

Ex 1: \( \frac{1}{2}, 1, \sqrt{2} \) \( \frac{1}{2}, 1 + \sqrt{2}, 1 - \sqrt{2} \) \in \( \mathbb{Q} (\sqrt{2}) \)

Prop 1: If \( \{ e_1, \ldots, e_n \} \) are bases for \( V \) then \( m = n \).

Def 1: If \( \{ e_1, \ldots, e_n \} \) is a basis for a vec. space \( V \), \( \text{dim}(V) = n \).
Theorem 20.15. Let $V$ be a vector space of dimension $n$.

1. If $S = \{v_1, \ldots, v_n\}$ is a set of linearly independent vectors for $V$, then $S$ is a basis for $V$.

2. If $S = \{v_1, \ldots, v_n\}$ spans $V$, then $S$ is a basis for $V$.

3. If $S = \{v_1, \ldots, v_k\}$ is a set of linearly independent vectors for $V$ with $k < n$, then there exist vectors $v_{k+1}, \ldots, v_n$ such that

$$\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$$

is a basis for $V$.

Fields

- When is a field $F$ contained in a larger field $G$?
- What fields are between $\mathbb{Q}$ and $\mathbb{R}$?

Let $F$ be a field, $p(x) \in F[x]$:

Can we find a field $E$, $F \subseteq E$, such that $p(x)$ factors into linear factors over $E[x]$?

**Example**

Consider $p(x) = x^4 - 5x^2 + 6 \in \mathbb{Q}[x]$,

$$p(x) = (x^2 - 2)(x^2 - 3)$$

- $p$ has no zeros in $\mathbb{Q}$, has 4 zeros in $\mathbb{IR}$
- can find smaller fields where $p(x)$ has zeros:
  - Extension field
    $$\mathbb{Q}(\sqrt{2}) = \left\{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \right\}$$
    $$\mathbb{Q}(\sqrt{3}) = \left\{ a + b\sqrt{3} \mid a, b \in \mathbb{Q} \right\}$$
- 2 roots in neither field.
Extension Fields

A field $E$ is an extension field of a field $F$ if $F$ is a subfield of $E$. $F$ is called the base of $E$. Write $F \subset E$.

Example 1: $F = \mathbb{Q}(\sqrt{2}) = \{ a + b \sqrt{2} \mid a, b \in \mathbb{Q} \}$

$E = \mathbb{Q}(\sqrt{2} + \sqrt{3}) = \{ a + b(\sqrt{2} + \sqrt{3}) \mid a, b \in \mathbb{Q} \}$

$E$ is an extension field of $F$:

$\sqrt{2} + \sqrt{3} \in E \implies \frac{1}{\sqrt{2} + \sqrt{3}} = \sqrt{3} - \sqrt{2} \in E$

Note we now have an extension field with all roots of $x^4 - 5x^2 + 6$

Aside:

Field of fractions of $\mathbb{Z}[\sqrt{n}] \cong \mathbb{Q}(\sqrt{n})$

Any element of field of fractions of $\mathbb{Z}[\sqrt{3}]$

\[
\frac{c + d\sqrt{3}}{e + f\sqrt{3}} \rightarrow \frac{c + b\sqrt{3}}{e + f\sqrt{3}} = \text{conjugation, simplify.}
\]

Homework problem