Feild of fractions

D - integral domain

$$S = \{(a,b) \mid a, b \in D, b \neq 0\}$$

Define an eq. relation

$$(a,b) \sim (c,d) \iff ad = bc \text{ in } D$$

$$\left(\text{Think } \frac{a}{b} = \frac{c}{d}\right)$$

Lemma $\sim$ above is an equivilence relation.

proof:

- $\sim$ reflexive since $ab = ba$ since commutive
  $(a,b) \sim (a,b)$

(Symetric): $(a,b) \sim (c,d) \iff (c,d) \sim (a,b)$

$\Downarrow$

$ad = bc$ sames $cb = da$

(Transitive): $(a,b) \sim (c,d)$, $(c,d) \sim (e,f)$

$ad = bc \quad \cdot f \qquad cf = de$

$$afd = adf = bcf = bde = bed$$

Since D is an int. dom, can cancel

$\Rightarrow af = be \Rightarrow (a,b) \sim (e,f)$

$\blacksquare$

$\therefore$ S is a set of equivalence classes

Write $F_D = S$ ← feild of fractions of D.

$a, b, c, d \in D$

$$[a, b] + [c, d] = [ad + bc, bd]$$

numerator denominator

$$[a, b] \cdot [c, d] = [ac, bd]$$

Lemma | Operations in F_D (above) are well-defined.

Proof: (Addition) Suppose $[a_1, b_1] = [a_2, b_2]$, $[c_1, d_1] = [c_2, d_2]$

Show $[a_1, b_1] + [c_1, d_1] = [a_1 d_1 + b_1 c_1, b_1 d_1] = [a_2, b_2] + [c_2, d_2]$

$= [a_2 d_2 + b_2 c_2, b_2 d_2]$

Show $(a_1 d_1 + b_1 c_1) b_2 d_2 = (a_2 d_2 + b_2 c_2)(b_1 d_1)$

$a_1 b_2 = b_1 a_2$ $\quad$ $c_1 d_2 = d_1 c_2$

$$\begin{aligned}(a_1 d_1 + b_1 c_1) b_2 d_2 &= a_1 d_1 b_2 d_2 + b_1 c_1 b_2 d_2 \\ &= b_1 a_2 d_1 d_2 + b_1 b_2 d_1 c_2 \\ &= (a_2 d_2 + b_2 c_2)(b_1 d_1)\end{aligned}$$

$\boxtimes$

Lemma) F_D with eq. rel. $\sim$ and ops.

$$[a,b] + [c,d] = [ad+bc,\ bd]$$

numerator   denominator

$$[a,b]\cdot[c,d] = [ac,\ bd]$$

is a field. $(a,b \in D)$ integral domain.

Proof:

Add identity: $[0,1] = \frac{0}{1} \Rightarrow [a,b] + [0,1] = [a\cdot 1 + b\cdot 0,\ b\cdot 1]$
$= [a,b]$

$[0,a] = [0,1]$

$0\cdot a = 0\cdot 1$

mult. id: $[1,1]$

Add. inverse is $[-a,b]$

mult inverse is $[b,a]$

$ab = ab \Leftrightarrow 1 = 1$

$$[b,a]\cdot[a,b] = [ab,ab] = [1,1]$$

etc.

$\blacksquare$

Thm) Let D be an int. domain. D can be imbedded in a field of fractions F_D where any $[a,b] \in F_D$ can be expressed as a quotient of 2 ele. of D

$$\boxed{[a,b] = \frac{[a,1]}{[b,1]},\quad a,b \in D.}$$

Also, F_D is unique, i.e. if E is any field s.t. $D \subseteq E$

then $\exists$ $\psi: F_D \to E$
$[a,b] \mapsto ab^{-1}$

giving an isomorphism $F_D \cong$ subfeild of E

Aside:

In practice write $\frac{a}{b} \in F_D$
"$[a,b]$"

subfeild = subring which is a feild

Think $E = \mathbb{R}$, or $\mathbb{C}$, $F_D = \mathbb{Q}$, $D = \mathbb{Z}$.

Proof:

First show D can be embedded in F_D

Define a map $\phi : D \to F_D$

$a \to [a,1] =$ "$\frac{a}{1}$"

Let $a,b \in D$

ϕ is a homomorphism:

$$\phi(a+b) = [a+b, 1] = [a,1] + [b,1] = \phi(a) + \phi(b)$$

$$\phi(ab) = [ab,1] = [a,1][b,1] = \phi(a)\phi(b)$$

$\therefore$ ϕ is a hom.

ϕ 1-1: suppose $\phi(a) = \phi(b)$

$$[a,1] = [b,1] \Rightarrow a = 1 \cdot a = 1 \cdot b = b$$

$\therefore$ D can be imbedded in F_D i.e. $D \cong \phi(D) \leq F_D$ ← First isomorphism theorem

Since $\ker(\phi) = \{0\}$ (since ϕ 1-1)

$\phi(D)$ is a subring of F_D

• Any $[a,b] \in F_D$ is a quotient (i.e. quo of two things in $\phi(D)$)

Since:

$$\phi(a)[\phi(b)]^{-1} = [a,1][b,1]^{-1} = [a,1][1,b]$$
$$= [a,b]$$

$$\therefore \; [a,b] = \frac{[a,1]}{[b,1]}$$

$\therefore$ Now Let E be a feild , $D \subseteq E$ (as a subring)

$$\psi : F_D \to E$$
$$[a,b] \mapsto ab^{-1}$$

• Show ψ is well defined. Suppose $[a_1, b_1] = [a_2, b_2]$

then $a_1 b_2 = b_1 a_2 \implies \psi([a_1, b_1]) = a_1 b_1^{-1} = a_2 b_2^{-1}$
$$= \psi([a_2, b_2])$$

$\therefore$ ψ is well defined.

• Show ψ is a hom. $[a,b], [c,d] \in F_D$

$$\psi([a,b] + [c,d]) = \psi([ad+bc, bd])$$
$$= (ad+bc)(bd)^{-1}$$
$$= ab^{-1} + cd^{-1}$$
$$= \psi([a,b]) + \psi([c,d])$$

$$\psi([a,b]\cdot[c,d]) = \psi([ac, bd]) = ac(bd)^{-1} = ab^{-1}cd^{-1} = \psi([a,b])\cdot\psi([c,d])$$

$\therefore$ ψ is a hom

Consider $\ker \psi$

If $\psi([a,b]) = ab^{-1} = 0 \implies a = b \cdot 0$

$\implies [a,b] = [0,b]$
$= [0,1]$

$\therefore$ $\ker \psi = [0,1]$ in F_D $\therefore$ ψ is 1-1

$\therefore$ by the isomorphism theorem

a subfeild of E.

$$F_D / \ker \phi = F_D \cong \psi(F_D) \subseteq E$$

$\boxtimes$

Ex) $\mathbb{Q}[x]$ - is an integral domain

$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; q(x) \neq 0,\ p(x), q(x) \in \mathbb{Q}[x] \right\}$$

Ex) $\mathbb{Q}(\sqrt{2})$ contains $\mathbb{Z}$

and $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$

coro) Let F be a feild of characteristic zero

Then F contains a subfeild isomorphic to $\mathbb{Q}$

coro) Let F be a feild of charcteristic p.

Then F contains a sub field isomorphic to $\mathbb{Z}_p$.

$\mathbb{Q}[x]$, , or $\mathbb{Z}_p[x]$, $\mathbb{Z}_p[i]$ etc.

Vector spaces

Can define a vector space over any feild F.

Def] vector space V over a feild F is:

- Abelian group (with addition) with a scalar product αv for $\alpha \in F$, $v \in V$ s.t. $(\alpha, \beta \in F, v, w \in V)$
  - $\alpha(\beta v) = (\alpha\beta)v$
  - $(\alpha+\beta)v = \alpha v + \beta v$
  - $\alpha(v+w) = \alpha v + \alpha w$
  - $1 \cdot v = v$

$\vdots$

Ex] $\mathbb{R}^n$, $\mathbb{C}^n$

Ex] If F is a feild, $F[x]$ is a vector space over F

- the vectors in $F[x]$ are polynomials
  - vector add is poly add.
  - $\alpha f(x)$
    scalar mult.

Ex] $C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ continuous} \}$

Ex] $V = \mathbb{Q}(\sqrt{2})$ is a v. space over $\mathbb{Q}$

$$u+v = (a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$$

Propositions Let V be a v. space over F. The following holds:

- $0v = 0 \in V \quad \forall v \in V,\ 0 \in F$
- $\alpha 0 = 0 \quad \forall \alpha \in F \quad 0 \in V$
- if $\alpha v = 0 \Rightarrow \alpha = 0 \in F$ or $v = 0 \in V$
- $(-1)v = -v$
- $-(\alpha v) = (-\alpha)v = \alpha(-v)$.

Subspaces

W is a subspace of a v. space V is W is closed under vector add. and scalar mult. i.e

- $\alpha w \in W \quad \forall \alpha \in F,\ w \in W$
- $w + v \in W \quad \forall w, v \in W$.

Ex

$$W = \{\text{poly. in } F[x] \text{ with no odd powers}\}$$
$$= \left\{ \sum_{i=0}^{n} a_i x^{2i} \;\middle|\; n \in \mathbb{Z},\ a_i \in F \right\}$$

is a subspace of $F[x]$.

Def $v_1, \dots, v_n \in V$, $\alpha_1, \dots, \alpha_n \in F$

$$w = \sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

w is a <u>linear combination</u> of $v_1, \dots, v_n$

$$W = \mathrm{Span}_F(v_1, \dots, v_n) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; \forall \alpha_i \in F \right\}$$

$\uparrow$

spanned by $v_1, \dots, v_n$ over F.