An Algorithm to Compute the Chern-Schwartz-Macpherson Classes and Euler Characteristic of a Complete Intersection

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c_{SM} Classes for Complete Intersections

- Let V be a closed possibly singular scheme-theoretic global complete intersection subscheme of $\mathbb{P}^n$.
- Using a result of Fullwood [6] we develop a probabilistic algorithm to compute the Chern-Schwartz-MacPherson class (c_{SM}) and Euler characteristic of V.
- This algorithm complements existing algorithms by providing performance improvements in the computation of the c_{SM} class and Euler characteristic for complete intersection schemes where the intersection of most of the generators is smooth.
- The algorithms can be implemented symbolically using Gröbner bases calculations, or numerically using homotopy continuation.

The topological Euler characteristic

- The Euler characteristic χ is an important topological invariant which allows for the categorization of topological spaces.
- Has numerous applications, for example it is applied to problems of maximum likelihood estimation in algebraic statistics by Huh [10] and to string theory in physics by Collinucci et al. [5] and by Aluffi and Esole [3].
- For projective schemes it can be computed several different ways, for example from Hodge numbers, or as we do here, from the Chern-Schwartz-Macperhson class.
- In particular from $c_{SM}(V)$ we may immediately obtain the Euler characteristic of V, $\chi(V)$ using the well-known relation which states that $\chi(V)$ is equal to the degree of the zero dimensional component of $c_{SM}(V)$.

Chern-Schwartz-MacPherson Classes

- The Chow ring of $\mathbb{P}^n$ is $\oplus_{j=0}^{n} A^j(\mathbb{P}^n)$ where $A^j(\mathbb{P}^n)$ is the group of codimension j-cycles modulo rational equivalence.
- We have that $A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})$ where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is the class of a hyperplane in $\mathbb{P}^n$, so that a hypersurface W of degree d will be represented by $[W] = d \cdot h$ in $A^*(\mathbb{P}^n)$.
- We consider the c_{SM} class (and other characteristic classes) as elements of $A^*(\mathbb{P}^n)$.
- The c_{SM} class generalizes the Chern class of the tangent bundle to singular varieties/schemes, i.e. $c(TV) \cap [V] = c_{SM}(V)$ when V is a smooth subscheme of $\mathbb{P}^n$.

Chern-Schwartz-MacPherson Classes

- The c_{SM} class has important functorial properties, in particular its relation to the Euler characteristic.
- When V is a subscheme of $\mathbb{P}^n$ the class $c_{SM}(V)$ can, in a sense, be thought of as a more refined version of the Euler characteristic since it in fact contains the Euler characteristics of V and those of general linear sections of V for each codimension.
- Specifically, if $\dim(V) = m$, starting from $c_{SM}(V)$ we may directly obtain the list of invariants

$$\chi(V), \chi(V \cap L_1), \chi(V \cap L_1 \cap L_2), \dots, \chi(V \cap L_1 \cap \cdots \cap L_m)$$

where $L_1, \dots, L_m$ are general hyperplanes.

Example: c_{SM} Class and Euler Characteristics

- Consider the variety $V = V(x_0x_3 - x_1x_2)$ in $\mathbb{P}^3 = \mathrm{Proj}(k[x_0, \ldots, x_3])$ which is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.
- We may compute $c_{SM}(V) = 4h^3 + 4h^2 + 2h$ and obtain the Euler characteristics of the general linear sections using an involution formula given by Aluffi in [2], specifically:
  - First consider the polynomial $p(t) = 4 + 4t + 2t^2 \in \mathbb{Z}[t]/(t^4)$ given by the coefficients of the c_{SM} class above.
  - Next apply Aluffi's involution

$$p(t) \mapsto \mathcal{I}(p) := \frac{t \cdot p(-t-1) + p(0)}{t+1} = 2t^2 - 2t + 4.$$

- This gives $\chi(V) = 4$, $\chi(V \cap L_1) = 2$, and $\chi(V \cap L_1 \cap L_2) = 2$.

Previous Algorithms to Compute the c_{SM} Class

Consider the hypersurface $V(f) \subset \mathbb{P}^n$ defined by the homogeneous polynomial f.

- All methods employ Theorem 2.1 of Aluffi [1], which may be expressed as
$$c_{SM}(V(f)) = (1+h)^{n+1} - \sum_{j=0}^{n} g_j(-h)^j(1+h)^{n-j} \text{ in } A^*(\mathbb{P}^n).$$
- The differences in the methods lay in how the g_j's are understood and computed.
- The first algorithm to compute $c_{SM}(V(f))$ was that of Aluffi [1].
  - To compute the g_j's this algorithm requires the computation of the blowup of $\mathbb{P}^n$ along the singularity subscheme of $V(f)$.
  - The computation of such blowups can be an expensive operation, making this algorithm impractical for many examples.

Previous Algorithms to Compute the c_{SM} Class

- Another algorithm to compute the c_{SM} class of a hypersurface was given by Jost in [11].
  - This method finds the g_j's by computing the degrees of certain residual sets via a particular saturation.
- In [9] the author gives a method to compute the g_i's, and hence the class $c_{SM}(V(f))$, using the projective degree of a rational map defined by the polynomials $(\frac{df}{dx_0}, \ldots, \frac{df}{dx_n})$, this method is also probabilistic and provides a performance improvement in many cases.
- All these methods require the use of the inclusion-exclusion property of c_{SM} classes when the scheme $V \subset \mathbb{P}^n$ has codimension higher than one.

Inclusion/Exclusion for c_{SM} Classes

- Specifically for V_1, V_2 subschemes of $\mathbb{P}^n$ the inclusion-exclusion property for c_{SM} classes states

$$c_{SM}(V_1 \cap V_2) = c_{SM}(V_1) + c_{SM}(V_2) - c_{SM}(V_1 \cup V_2). \quad (1)$$

- Inclusion/Exclusion allows for the computation of $c_{SM}(V)$ for V of any codimension.
- This requires exponentially many c_{SM} computations relative to the number of generators of I.
- Must consider c_{SM} classes of products of many or all of the generators of I, which may have significantly higher degree than the original scheme V.

c_{SM} Classes Without Inclusion/Exclusion

- In certain special cases one may avoid using inclusion/exclusion.
- Begin by considering the following relation between the $c_{SM}(V)$ and the Chern-Fulton-Johnson ($c_{FJ}(V)$) and Milnor ($\mathcal{M}(V)$) classes of V,

$$c_{SM}(V) = c_{FJ}(V) - (-1)^{\mathrm{codim}(V)}\mathcal{M}(V). \tag{2}$$

- When V is a global complete intersection there is a known formula for $c_{FJ}(V)$ in terms of the degrees of the defining equations of V.
- In [6] Fullwood gives an expression for the Milnor class of a global complete intersection $V = V(f_0, \dots, f_r)$ where $V(f_0, \dots, f_{r-1})$ is smooth (scheme theoretically).

c_{SM} Classes Without Inclusion/Exclusion

Combining the result of Fullwood with the relation between the c_{SM} and Milnor classes we may prove the following theorem.

Theorem 1

Let $I = (f_0, \ldots, f_r)$ be a homogeneous ideal in $k[x_0, \ldots, x_n]$ with k an algebraically closed field of characteristic zero and set $\deg(f_i) = d_i$. For $V(f_0, \ldots, f_r)$ a globally complete intersection subscheme of $\mathbb{P}^n$ such that $V(f_0, \ldots, f_{r-1})$ is smooth (scheme theoretically) we have:

$$c_{SM}(V) = (1+h)^{n+1} \cdot \prod_{i=0}^{\mathrm{codim}(V)} \frac{d_i h}{1+d_i h} -$$

$$\frac{(-1)^r(1+h)^{n+1}}{\prod_{i=0}^{r}(1+d_i h)} \left(\sum_{p=0}^{r} h^p \sum_{i=0}^{p} \binom{r-i}{p-i} (-1)^i (d_r)^{p-i} \cdot \tilde{c}_i \right) \cdot \left(\sum_{i=0}^{n} \frac{(-1)^i s_i h^i}{(1+d_r)^i} \right)$$

in $A_*(\mathbb{P}^n)$.

c_{SM} Classes Without Inclusion/Exclusion

- Above s_i and $\tilde{c}_i$ are

$$\prod_{i=0}^{r}(1+d_i h)=\sum_{i=0}^{r}\tilde{c}_i h^i, \quad s(Y,\mathbb{P}^n)=\sum_{i=0}^{n} s_i h^i$$

  where Y is the singularity subscheme of X in $\mathbb{P}^n$, that is the zero scheme of the ideal defined by the appropriate minors of the Jacobian of I.
- The only unknown in the expression is the Segre class $s(Y,\mathbb{P}^n)$.
- Hence to obtain an algorithm to compute c_{SM} classes in this case we may combine the result of Theorem 1 with the method to compute Segre classes using the projective degree of a rational map given by the author in [9].

Projective Degrees and the Segre class

- Consider a rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$. In the manner of Harris (Example 19.4 of [8]) we may define the *projective degrees* of the map ϕ as a list of integers $(g_0, \ldots, g_n)$ where $g_i = \mathrm{card}\left(\phi^{-1}\left(\mathbb{P}^{m-i}\right) \cap \mathbb{P}^i\right)$.
  - Here $\mathbb{P}^{m-i} \subset \mathbb{P}^m$ and $\mathbb{P}^i \subset \mathbb{P}^n$ are general hyperplanes of dimension $m-i$ and i respectively and card is the cardinality of a zero dimensional set.
- Let $J = (w_0, \ldots, w_m) \subset R = k[x_0, \ldots, x_n]$ be a homogeneous ideal defining a scheme $Y = V(J) \subset \mathbb{P}^n$ and let $h = c_1\left(\mathcal{O}_{\mathbb{P}^n}(1)\right)$ be the hyperplane class in $A_*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})$.
- Since J is homogeneous we may assume that the $\deg(w_i) = d$ for all i.

Projective Degrees and the Segre class

- Also let $(g_0, \dots, g_n)$ be the projective degrees of the map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $\phi : p \mapsto (w_0(p) : \cdots : w_m(p))$. By Proposition 3.1 of Aluffi [1] we have

$$s(Y, \mathbb{P}^n) = 1 - \sum_{i=0}^{n} \frac{g_i h^i}{(1+dh)^{i+1}} \in A_*(\mathbb{P}^n). \quad (3)$$

- To compute the projective degrees g_i above we may apply [9, Theorem 3.1] which allows us to compute each g_i by computing the degree of either a certain zero dimensional ideal or a certain saturation, note this computation is probabilistic and yields the correct result for a choice of objects lying in an open dense Zariski set.

A Direct Algorithm to Compute the c_{SM} Class

- **Input:** a homogeneous ideal $I = (f_0, \ldots, f_r)$ in $k[x_0, \ldots, x_n]$ defining a complete intersection scheme $V = V(I) \subset \mathbb{P}^n$ such that $V(f_0, \ldots, f_{r-1})$ is smooth (scheme theoretically).
- **Output** $c_{SM}(V)$ and/or $\chi(V)$.
    - Let K be the $(r+1) \times (r+1)$ minors of the Jacobian matrix of I, compute $J = (K + I) : (x_0, \ldots, x_n)^\infty$ to obtain the singularity subscheme $Y = V(J)$, of V.
    - Apply [9, Theorem 3.1] with the map defined by the ideal J to compute the projective degrees $g_0, \ldots, g_n$.
    - Compute $s(Y, \mathbb{P}^n)$ by using (3) and the projective degrees $g_0, \ldots, g_n$ computed above.
    - Apply Theorem 1 to obtain $c_{SM}(V)$.

A Direct Algorithm to Compute the c_{SM} Class

- The main computational cost of the algorithm described above is the computation of the projective degrees $g_0, \dots, g_n$.
- This computation consists of finding the degree of the zero dimensional ideal described in [9, Theorem 3.1], or by computing a certain saturation.
- This can be accomplished symbolically using Gröbner bases calculations, or numerically using homotopy continuation via a package such as Bertini [4].

Extending the Algorithm to any Complete Intersection

- For a complete intersection $V = Z \cap X_1 \cap X_2$ with $Z \subset \mathbb{P}^n$ smooth (scheme-theoretically) and $X_1 = V(f_1)$, $X_2 = V(f_2)$ singular hypersurfaces in $\mathbb{P}^n$ we can show

$$c_{SM}(V) = c_{SM}(Z \cap X_1) + c_{SM}(Z \cap X_2) - c_{SM}(Z \cap (X_1 \cup X_2)).$$

- This result allows us to extend the application of Theorem 1 to complete intersections $V = V(I)$ where several of the generators of the ideal I are singular.
- If only a few of the generators are singular this could offer a computational speed boost.
- At worst, when all of the generators are singular, this will reduce to inclusion/exclusion.

Running Time Comparison for c_{SM} Algorithm

- Running times of the direct algorithm discussed here compared to several other algorithms which use inclusion-exclusion.
  - All methods are implemented in Macaulay2 [7] and preformed over $\mathbb{Q}$.
  - Those that took longer than 600s are denoted with -.
  - For the methods csm_dir (Th. 1) and csm_I_E ([9]) the timings in ( ) indicate symbolic computation using a saturation.
  - Timings in [ ] indicate numeric computations using Bertini [4].

Input	CSM (Aluffi)	CSM ([11])	csm_dir (Th. 1)	csm_I_E ([9])
$V_1 \subset \mathbb{P}^7$	-	- [-]	0.3s (0.2s) [4.8s]	- (116.5s) [-]
$V_2 \subset \mathbb{P}^4$	-	1.7s [-]	0.3s (0.1s) [1.3s]	1.2s (1.2s) [44.1s]
$V_3 \subset \mathbb{P}^6$	-	27.7s [-]	7.2s (2.2s) [-]	33.2s (53.2s) [-]
$V_4 \subset \mathbb{P}^5$	-	- [-]	4.6s (0.7s) [5.5s]	- (-) [-]
$V_5 \subset \mathbb{P}^6$	-	- [-]	19.9s (7.9s) [24.9s]	- (-) [-]

Running Time Comparison for c_{SM} Algorithm

In the table above

$$V_1 = V(\tfrac{13}{21}x_0^2 + \tfrac{7}{5}x_1^2 - \tfrac{24}{7}x_2^2 + 13x_3^2 + 8x_4^2 - \tfrac{17}{6}x_5^2 + 2x_6^2 + \tfrac{14}{11}x_7^2, x_1^2x_5 - x_0^2x_4),$$

$$V_2 = V(3x_0^2 + 19x_1^2 + \tfrac{7}{8}x_2^2 + 12x_3^2 + 13x_4^2, \tfrac{3}{4}x_0 + \tfrac{9}{5}x_1 + \tfrac{6}{19}x_2 + \tfrac{12}{7}x_3 - 15x_4, x_0^2 - x_4^2),$$

$$V_3 = V(3x_0^2 + 19x_1^2 + \tfrac{7}{8}x_2^2 + 12x_3^2 + 9x_4^2 + \tfrac{7}{3}x_5^2 + \tfrac{2}{5}x_6^2, x_2^3x_3 - x_3x_5^3)\ ,$$

$$V_4 = V(5x_0^2+9x_1^2+\tfrac{7}{9}x_2^2+2x_3^2+\tfrac{3}{5}x_4^2+\tfrac{7}{3}x_5^2, 23x_0+9x_1+7x_2+2x_3+4x_4+3x_5, x_2x_0x_3-x_3x_5x_4)$$

and

$$V_5 = V(\tfrac{3}{2}x_0^2+\tfrac{17}{9}x_1^2-\tfrac{4}{7}x_2^2+3x_3^2+38x_4^2-\tfrac{72}{7}x_5^2+12x_6^2, x_0x_6-x_0^2, \tfrac{4}{3}x_0^2+\tfrac{5}{2}x_0x_1+\tfrac{9}{4}x_1^2+\tfrac{1}{5}x_0x_2+ \tfrac{1}{3}x_1x_2 + x_2^2 + x_0x_3 + 4x_1x_3 + \tfrac{9}{8}x_2x_3 + x_3^2 + x_0x_4 + \tfrac{7}{4}x_1x_4 + \tfrac{1}{3}x_2x_4 + \tfrac{7}{3}x_3x_4 + \tfrac{2}{3}x_4^2 + \tfrac{1}{2}x_0x_5 + \tfrac{1}{2}x_1x_5+x_2x_5+\tfrac{9}{8}x_3x_5+\tfrac{9}{2}x_4x_5+\tfrac{1}{2}x_5^2+\tfrac{3}{5}x_0x_6+\tfrac{10}{3}x_1x_6+\tfrac{3}{8}x_2x_6+x_3x_6+\tfrac{1}{6}x_4x_6+2x_5x_6+\tfrac{9}{5}x_6^2).$$

Direct c_{SM} Algorithm Discussion

- It seems that for the types of examples for which the result of Theorem 1 is applicable it offers a performance increase over the algorithms which use inclusion-exclusion.
- As previously observed by the author in [9] the symbolic versions of all algorithms are slightly faster than the numeric versions; the reason for this is unclear to us.
- Regarding the symbolic approaches, for our new method, computing saturations appears to be the faster solution in most cases, at least over $\mathbb{Q}$.
- Both methods apply as well over finite fields, with substantially improved running times; in that case, using saturations is slower than computing the degree of the zero dimensional ideal described in [9, Theorem 3.1] for most cases.

Future/Current Work

- Obtain running time bounds for the algorithm presented here.
- Try using the algorithm on examples arising from applications and see how it preforms there.
- Investigate possible improvements to the M2 implementation of the algorithm.

Thank you for listening!

References (1)

Paolo Aluffi.
Computing characteristic classes of projective schemes.
Journal of Symbolic Computation, 35(1):3–19, 2003.

Paolo Aluffi.
Euler characteristics of general linear sections and polynomial Chern classes.
Rendiconti del Circolo Matematico di Palermo, pages 1–24, 2013.

Paolo Aluffi and Mboyo Esole.
New orientifold weak coupling limits in F-theory.
Journal of High Energy Physics, 2010(2):1–53, 2010.

Daniel J. Bates, Jonathan D. Hauenstein, Andrew J. Sommese, and Charles W. Wampler.
Bertini: Software for numerical algebraic geometry.

Andres Collinucci, Frederik Denef, and Mboyo Esole.
D-brane deconstructions in IIB orientifolds.
Journal of High Energy Physics, 2009(02):005, 2009.

James Fullwood.
On Milnor classes via invariants of singular subschemes.
Journal of Singularities, 8:1–10, 2014.

Daniel R. Grayson and Michael E. Stillman.
Macaulay2, a software system for research in algebraic geometry, 2013.

References (2)

Joe Harris.
Algebraic geometry: a first course, volume 133.
Springer, 1992.

Martin Helmer.
An algorithm to compute the topological Euler characteristic, Chern-Schwartz-Macpherson class and Segre class of projective varieties.
arXiv:1402.2930, 2014.

June Huh.
The maximum likelihood degree of a very affine variety.
Compositio Mathematica, pages 1–22, 2012.

Christine Jost.
An algorithm for computing the topological Euler characteristic of complex projective varieties.
arXiv:1301.4128, 2013.