A phase transition in a random loop model on infinite trees

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Spatial Random Permutations
Quick recap of permutations

- Finite permutations decompose as a product of disjoint cycles.
- Infinite permutations do too, but the cycles may be infinite.

Example:

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(i)$</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Using cycle notation, $\sigma$ can be decomposed as $(1 \ 3)(2 \ 5 \ 4)$. 
What is a spatial random permutation?

A “spatial random permutation” vaguely refers to a random permutation model whose index set possesses some spatial or geometric structure which affects the permutation structure.

For us, this will be via the index set being the vertices of a graph.
Tóth’s model

Introduced in [Tóth ’93] to study the quantum Heisenberg ferromagnet:

- Let $\Lambda \subseteq \mathbb{Z}^d$ be a finite box.
- Let $\mu_T$ be a measure on permutations on the vertices of $\Lambda$.
- Define $\nu_T$ by

$$d\nu_T(\sigma) = \frac{1}{Z} \cdot 2^{\#\text{cycles}(\sigma)} d\mu_T(\sigma),$$

i.e. reweight the permutations by the number of cycles present and normalize.
Tóth’s model

Let $\sigma_\Lambda$ be a sample from $\nu_T$.

Very roughly speaking, Tóth showed that, if $\Lambda \rightarrow \mathbb{Z}^d$, then there is a correspondence (with $T = \text{time/inverse temperature}$):

appearance of macrocycles in $\sigma_\Lambda$ $\leftrightarrow$ physical phase transition in the spin-1/2 $q$-Heisenberg ferromagnet.

This is a recurring theme with spatial random permutation models connected to physical models. So we want to prove macrocycles exist.
Consider a graph $G = (V, E)$. The random stirring process (RSP) is a process of permutations on $V$: $(\sigma_t)_{t \geq 0}$, with $\sigma_0 = \text{Id}$.

- To each $e \in E$, associate an independent rate 1 Poisson clock.
- Suppose $e = \{u, v\}$ rings at time $t$. Left compose $\sigma_{t-}$ with $(u \ v)$:
  \[ \sigma_t = (u \ v) \circ \sigma_{t-}, \]
  so we maintain right-continuity.
An example

Suppose \( \{v_1, v_2\} \) rings at \( t = 1/2 \) and \( \{v_1, v_3\} \) rings at \( t = 1 \).

Then \( \sigma_t \) is

\[
\sigma_t = \begin{cases}
\text{Id} & 0 \leq t < \frac{1}{2} \\
(v_1 \ v_2) & \frac{1}{2} \leq t < 1 \\
(v_1 \ v_2 \ v_3) & t \geq 1.
\end{cases}
\]

\[
[(v_1 \ v_3) \circ (v_1 \ v_2) = (v_1 \ v_2 \ v_3)].
\]
To know $\sigma_{5/4}(v_2)$, place a particle at the vertex $v_2$, and let it move upwards at unit speed. When it hits a cross, it jumps over instantly and continues motion up.

The vertex the particle is at at time $t = 5/4$ is exactly $\sigma_{5/4}(v_2)$; in this case, $v_3$. 

\[ t = \frac{5}{4} \]
\[ t = 1 \]
\[ t = \frac{1}{2} \]
Another view

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$\begin{align*}
    t &= \frac{5}{4} \\
    t &= 1 \\
    t &= \frac{1}{2}
\end{align*}$

$\begin{array}{c}
v_2 \\
v_1 \\
v_3
\end{array}$
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We can take this viewpoint in general. Fix $T$. To study $\sigma_T$, we study a related process, called the cyclic time random meander of parameter $T$: $\text{CyTRM}(T)$. Defined on $(0, \infty)$.

For a graph $G = (V, E)$, associate to each $e \in E$ an independent rate 1 Poisson point process on $[0, T)$. These are the crosses from earlier.

Visualize a vertical pole of height $T$ at each $v \in V$. Poles are connected by crosses at the points of the point processes.
Let \( T = \frac{5}{4} \) started at a vertex \( v \). It is defined on \([0, \infty)\), with \( X(0) = v \).

The motion is as before, except when we reach the top of a pole.
The Cyclic Time Random Meander (CyTRM)

Let $X = \text{CyTRM}(T)$ started at a vertex $v$. It is defined on $[0, \infty)$, with $X(0) = v$.

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When the particle reaches the top of a pole, it instantly appears at the bottom and continues motion: it is *cyclic*. 

\[ T = \frac{5}{4} \]
The Cyclic Time Random Meander (CyTRM)

When the particle reaches the top of a pole, it instantly appears at the bottom and continues motion: it is \textit{cyclic}.

Thus \( X(T) = \sigma_T(v) \).

Here, with \( T = \frac{5}{4} \), \( X\left(\frac{5}{4}\right) = v_3 \).
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By the cyclic nature,
$X(2T) = \sigma_T^2(v)$.
Here, $X\left(\frac{5}{2}\right) = v_1$. 

$T = \frac{5}{4}$
Similarly, if $X$ is started at $\nu$, 

$$X(kT) = \sigma_T^k(\nu).$$

So if $\nu$ lies in an infinite cycle in $\sigma_T$, $X(kT) \neq \nu$ for any $k$.

The logic can be extended to say that

$$\text{transience of CyTRM}(T) \iff \nu \in \text{infinite cycle in } \sigma_T \text{ started at } \nu$$

We want to analyse transience of CyTRM$(T)$ as a function of $T$. 
The Actual Model

Introduced by Ueltschi in [Ueltschi ’13]. Instead of just crosses, we also have *double bars*: when the particle encounters a double bar, it jumps over instantly, but its direction of motion is *reversed*.
If the particle hits the bottom while moving down, it cycles to the top instantly. In either case, the direction of motion is maintained.
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The Actual Model: $\text{CyTRM}(u, T)$

- Collectively, crosses and double bars will be called *bridges*.
- New parameter $u \in [0, 1]$ : probability that a bridge is a cross. Otherwise, a double bar.
- The $u = 1$ case was the original model first described.
- Denote the modified model $\text{CyTRM}(u, T)$.
- As before, our interest is in *transience* of this process.
### Theorem

1. Let $G$ be a rooted tree of bounded degree with at least $d_0$ offspring at every vertex. Then there exists a $T_0$ such that $\text{CyTRM}(u, T)$ is transient when $T > T_0$. We may take $T_0 = 0.495$ and $d_0 = 16$.

2. If $G$ has exactly $d$ offspring at every vertex, then there exists $T_c(u, d)$ such that $\text{CyTRM}(u, T)$ is transient for $T > T_c$ and recurrent for $0 < T < T_c$.

Asymptotic formula for $T_c$ from [Björnberg-Ueltschi ’18]:

$$T_c(u, d) = \frac{1}{d} + \frac{1 - u(1 - u) - \frac{1}{6}(1 - u)^2}{d^2} + o(d^{-2}).$$
Plot of $1 - u(1 - u) - \frac{1}{6}(1 - u)^2$
Previous Work
Let $G$ be the regular tree of offspring number $d$.

The percolation probability for $G$ is $d^{-1}$. If $T$ is such that the probability of at least one bridge on an edge is less than $d^{-1}$, we have recurrence.

So we have recurrence for $T < T_{\text{perc}} := \log \frac{d}{d-1} = \frac{1}{d} + \frac{1}{2d^2} + o(d^{-2})$.

(by equating $1 - e^{-T}$ with $d^{-1}$)
Previous work: Angel ’03

- $u = 1$ case studied on infinite regular trees; $u \in [0, 1]$ easily adapted.
- Established transience in a finite interval slightly above $T_c(1, d)$:
  $[d^{-1} + 2d^{-2}, \frac{1}{2}]$
- Outline: Identified a local configuration which forces transience, and showed that vertices with the local configuration form a Galton-Watson tree. In the mentioned interval, the GW mean offspring number is greater than 1.
- The local configuration requires a small number of bridges, which is unlikely for $T$ high; the argument works only for low $T$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Phase diagram for $d = 3$.}
\end{figure}
Previous work: Hammond I

- Only $u = 1$ case.
- Establishes a similar result as ours: there exists $T_0(= 429d^{-1})$ such that for sufficiently large $d$ and $T > T_0$, CyTRM(1, $T$) is transient.
- A large part of our work is extending and simplifying this argument; will speak more later.
Previous work: Hammond II

- Only $u = 1$ case, but also applies to $u \in [0, 1]$, as observed in [Björnberg-Ueltschi '18].
- Establishes monotonicity in a small interval around critical point: if $d^{-1} < T < T' < d^{-1} + 2d^{-2}$ and CyTRM($u, T$) is transient, then so is CyTRM($u, T'$).

![Diagram showing percolation and Angel phases for Hammond I and Hammond II phases](image-url)
Previous Work

[Björnberg-Ueltschi ’18]

- Found an asymptotic expansion of $T_c$:

$$T_c(u, d) = \frac{1}{d} + \frac{1 - u(1 - u) - \frac{1}{6}(1 - u)^2}{d^2} + o(d^{-2}).$$

- They show transience for $T \in (T_c, \frac{1}{d} + \frac{A}{d^{1-2}}]$ for $d > d_0 = d_0(A)$
- But in principle transience may not hold for arbitrarily large $T$...
- Shows that $T_c(u, d) > T_{perc}$ asymptotically.
Björnberg-Ueltschi ’18 preprint

- Extend their asymptotic formula when cycles are reweighted by $\theta > 0$ (as in Tóth’s model, where $\theta = 2$).

$$T_c(u, \theta, d) = \frac{\theta}{d} + \frac{\theta \left[1 - \theta u(1 - u) - \frac{1}{6} \theta^2 (1 - u)^2\right]}{d^2} + o(d^{-2}).$$
Betz-Ehlert-Lees-Roth ’18 preprint

- Further the expansion of $T_c(u, d)$ to order 4 in $d^{-1}$.
- Obtain sharper bounds for $T_c$ for finite $d$.
  - The gap between $T_{\text{perc}}$ and $T_c$ is established for all $d \geq 3$. 
Proof Overview
Proof Overview

- We need to show that for high $T$, $\text{CyTRM}(u, T)$ escapes to infinity with positive probability.
- This is simple when $u = 1$ and "$T = \infty$": it’s just simple random walk on the tree.
- So why is SRW on a tree transient?
Simple Case: SRW on tree

- Uniformly positive probability $p$ of departing to new territory at each step—a “frontier departure”.
- Then it either never returns, or, if it returns, two possibilities:
  - moves to new territory again—an “acceptable return”.
  - moves back into old territory
- If not an acceptable return, positive probability of moving to new territory next time.
So the distance from the root stochastically dominates the following random walk on $\mathbb{Z}$:

\[
\frac{1}{d+1} \xrightarrow{} \frac{d}{d+1}
\]

\[- \ldots - \xrightarrow{} x - 1 \xrightarrow{} x \xrightarrow{} x + 1 \xrightarrow{} +\infty
\]

This has positive drift and so escapes to $+\infty$ with positive probability, which implies the original SRW is transient.
Now we don’t have complete independence. But after a frontier departure, we have some independence for duration $T$.

We introduce a proxy for the distance from the root: the number of “useful bridges” at time $t$.

Think of them as barriers the particle must undo to return to the root.

Main property of useful bridges: if an edge supports a useful bridge at time $t$, it has been crossed only once until that time.

$$\Rightarrow \ #\text{useful bridges} \leq \text{distance from root}.$$
We have to redefine an “acceptable return”:

A (first) return to a previously visited edge $e$ is acceptable if the particle then leaves to an unvisited vertex and moves forward consecutively $N$ times by duration $T$.

Note that the type of bridge crossed doesn’t matter, as long as the direction is away from the root.
We can lower bound probabilities of

(i) frontier departure
(ii) moving forward $N$ times in time $T$ given a frontier departure.

This gives a lower bound $p(N, T, d)$ for the probability of an acceptable return.

When a return is acceptable, gain $N - 2$ useful bridges at least. When not acceptable, lose 2 useful bridges at most.
Completing the argument

Looking at the number of useful bridges at suitable stopping times, it dominates the following random walk on $\mathbb{Z}$ with $p = p(N, T, d)$:

\[
\text{Drift} = N \times \frac{d - 1}{d + 1} \left(1 - e^{-(d+1)T/2}\right) \\
\times \left(1 - \frac{1}{d + 1}\right)^N \left[1 - e^{N-(d+1)T} \left(\frac{(d + 1)T}{N}\right)^N \right] - 2.
\]

Play with the parameters to make it positive $\Rightarrow$ transience.
Jakob Björnberg and Daniel Ueltschi (2018)
Critical parameter of random loop model on trees.

Balint Tóth (1993)
Improved lower bound on the thermodynamic pressure of
the spin 1/2 Heisenberg ferromagnet.

Daniel Ueltschi (2013)
Random loop representations for quantum spin systems.
Thank you