**Def.** Let \( A = (a_{ij})_{i,j \leq N} \) where \( a_{ij} \) are i.i.d \( \mathcal{CN}(0,1) \) and let

\[
X = \frac{1}{\sqrt{d}} (A + A^*)
\]

Then \( X \in \text{GUE}(N) \) (\( \mathcal{H}_N^{(a)} \) in Anderson).

**Def.** Let \( B = (b_{ij})_{i,j \leq N} \) where \( b_{ij} \) are i.i.d \( \mathcal{N}(0,1) \) and let

\[
Y = \frac{1}{\sqrt{d}} (B + B^t)
\]

Then \( Y \in \text{GOE}(N) \) (\( \mathcal{H}_N^{(a)} \) in Anderson).
Thm (Joint distribution of eigenvalues). Let $X \in \mathcal{H}_N^{(\beta)}$ be random with law $\rho N^{(\beta)}$ for $\beta = 1, 2$. The joint distribution of the eigenvalues $\lambda_1(X) \leq \cdots \leq \lambda_N(X)$ has the following density wrt the Lebesgue measure

$$N! \, \overline{C}_N^{(\beta)} \prod_{1 \leq i < j \leq N} |\Delta(x)|^\beta \prod_{i=1}^N \exp(-\beta x_i^2/4)$$

- Normalization constant
- Vandermonde determinant $\Delta(x) = \prod_{i<j} (x_j - x_i)$
A Jacobi matrix is a symmetric (Hermitian) tridiagonal matrix where the off-diagonal terms are strictly positive.

\[
X_N = \begin{bmatrix}
an_1 & b_1 & & \\
b_1 & \ddots & b_{n-1} & \\
 & \ddots & \ddots & b_{n-1} \\
 & & b_{n-1} & a_n
\end{bmatrix}
\]

We want to study the joint distribution of the eigenvalues of \( X \). To do so we will need to do the following,

1. Construct a map from \((a,b) \mapsto (\lambda, ?)\).

2. Compute the Jacobian of the above map.

3. Observe how different densities of \((a,b)\) affect the distribution of \( \lambda \).
Constructing the map.

$(a, b)$ has $2N-1$ variables, but we only have $N$ eigenvalues. So we must find $N-1$ auxiliary variables in order to create the map.

$v_1, \ldots, v_N$ orthonormal eigenvectors

$\lambda_1, \ldots, \lambda_N$ corresponding eigenvalues

Let $p_K = |\langle v_K, e_i \rangle|^2$ and they satisfy,

$$p_1 + \ldots + p_N = 1$$

so let $p_1, \ldots, p_{N-1}$ be the $N-1$ auxiliary variables.

Now we have enough variable to try to find a bijection. First let's identify $(\lambda, p)$ with the spectral measure of $X_N$,

$$v_X = \sum_{k=1}^{N} p_k \delta_{\lambda_k}$$
Now we can define the map $G: J_N \to M_N$ by

$$G(\lambda) = \nu_\lambda$$

where

$$J_N = \mathbb{R}^N \times \mathbb{R}^{N-1}_+$$

$$M_N = \mathbb{R}^N_+ \times \Delta_N$$

**Lemma.** Let $G$ be defined as above. Then,

(a) $G$ is a bijection

(b) \( \prod_{k=1}^{N-1} b_k(\lambda_k, \lambda_{k+1}) = \prod_{k=1}^{N} p_k \cdot \prod_{i<j} |\lambda_i - \lambda_j|^2 \)

(c) The Jacobian determinant of $G^{-1}$ is equal to (up to a sign),

$$J_{G^{-1}}(\lambda, p) = \frac{\prod_{k=1}^{N-1} b_k}{\prod_{k=1}^{N} |p_k|^2}$$
Proving the above lemma will be most of the work, so let’s hold off so we can understand some consequences first.

Let \((a,b)\) have joint density \(f(a,b)\) w.r.t the Lebesgue measure on \(\mathbb{T}_N\). Then the density of \((\lambda,p)\) on \(\mathbb{T}_N\) (w.r.t Lebesgue measure) is,

\[
g(\lambda,p) = f(a,b) \frac{\prod_{k=1}^{N-1} b_k}{2^{N-1} \prod_{k=1}^{N} p_k}
\]

What is a ‘nice’ choice of \(f\)?

\[
f(a,b) \propto \exp\left\{-\frac{\beta}{4} \left[ \sum_{k=1}^{N} a_k^2 + 2 \sum_{k=1}^{N-1} b_k^2 \right] \right\} \prod_{k=1}^{N-1} b_k^{\beta (N-k)-1}
\]

\[
= \text{Tr}(T^2)
\]

\[
= \sum_{k=1}^{N} \lambda_k^2
\]

Cancel parts of above and then use (b) of lemma

Then we get

\[
g(\lambda,p) \propto \exp\left\{-\frac{\beta}{4} \sum_i \lambda_i^2 \right\} \frac{\prod_{k=1}^{N-1} b_k^{\beta (N-k)}}{2^{N-1} \prod_{k=1}^{N} p_k} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_{k=1}^{N} p_k^{\beta - 1}
\]
Thm. Let the $a_k \sim N(0, \sigma)$ and $b_k \sim \mathcal{U}(n-k)$ (both independent and independent of each other). Then the eigenvalues of $X_N$ have density proportional to,

$$\exp\left\{ -\frac{\beta}{4} \sum_{k=1}^{n} \lambda_k^2 \right\} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

This density is called the $\beta$-Hermite ensemble.

What's next...

1. Show that Gaussian ensembles are $\beta$-Hermite ensembles.

2. (If time) Prove the lemma.
Reduction of GOE into Jacobi matrix.

\[ A \in \text{GOE}(N) \]

\[ A = \begin{bmatrix} a_i & v_i^t \\ v_i & B \end{bmatrix} \]

Want a \( P \in O(N-1) \) such that \( P v = r e_i \) and \( r = \|v\| \). One choice of \( P \) is,

\[ P = I - 2 uu^t, \quad u = \frac{v - r e_i}{\|v - r e_i\|} \]

\[ \hat{P} = \begin{bmatrix} 1 & 0^t \\ 0 & P \end{bmatrix} \]

\[ A_i = \hat{P} A \hat{P}^t = \begin{bmatrix} a_i & r e_i^t \\ r e_i & C \end{bmatrix} \]

Now let's repeat this process for the matrix \( C \).
\[
C = \begin{bmatrix}
  a_2 & v_2^t \\
  v_2 & D
\end{bmatrix} \quad \hat{Q}C\hat{Q}^t = \begin{bmatrix}
  a_d & r_{ae1}^t \\
  r_{ae1} & QDQ^t
\end{bmatrix}
\]

where \( r_a = \| v_a \| \). Repeating this procedure will result in the matrix

\[
\begin{bmatrix}
  a_1 & r_1 & & \\
  r_1 & & r_{N-1} & \\
  & \ddots & \ddots & r_N \\
  & & r_{N-1} & a_N
\end{bmatrix}
\]

Not only is it a Jacobi matrix, but it is \( \beta \)-Hermite.

\[
\begin{align*}
\text{GOE} & \mapsto \beta = 1 & \text{No corresponding matrices for general } \beta \\
\text{GUE} & \mapsto \beta = 2 \\
\text{GSE} & \mapsto \beta = 4
\end{align*}
\]
Proof of (b).

\( X_k \) = top-left \( k \times k \) submatrix of \( X_N \)

\( \hat{X}_k \) = bottom-right \( \cdots \) \( \cdots \) \( \cdots \) \( \cdots \)

\( \varphi_k \) = characteristic polynomial of \( X_k \)

\( \tilde{\varphi}_k \) = \( \cdots \) \( \cdots \) \( \cdots \) \( \cdots \) of \( \hat{X}_k \)

\( \lambda^{(k)}_j \) = eigenvalues of \( X_k \)

We have the following recursion,

\[ \varphi_k = (z - a_k) \varphi_{k-1} - b_{k-1}^2 \varphi_{k-2} , \quad \varphi_0 = 1 , \quad \varphi_{-1} = 0. \]

Fact: Eigenvalues of \( T_k \) are strictly interlaced with the eigenvalues of \( T_{k-1} \).

Let \( z = \lambda^{(k-1)}_j \) and multiply over all \( j \leq k-1 \).

\[ \prod_{j=1}^{k-1} \varphi_k(\lambda^{(k-1)}_j) = (-1)^{k-1} b_{k-1}^{2(k-1)} \prod_{j=1}^{k-1} \varphi_{k-2}(\lambda^{(k-1)}_j) \]
For \( \varphi_k \) and \( \varphi_{k-1} \) we have the relation,

\[
\prod_{j=1}^{k} \varphi_{k-1}(\lambda_j^{(k)}) = \pm \prod_{j=1}^{k-1} \varphi_k(\lambda_j^{(k-1)}) \\
= \pm b_{k-1}^{a(k-1)} \prod_{j=1}^{k-1} \varphi_{k-2}(\lambda_j^{(k-1)})
\]

Take the product over \( k \) and telescope \( k=2 \) to \( k=N \)

\[
\prod_{j=1}^{N} \varphi_{N-1}(\lambda_j) = \pm \prod_{j=1}^{N-1} b_j^{a_j}
\]

We can do the same procedure (in reverse) to the \( \tilde{\varphi}_k \)'s,

\[
\prod_{j=1}^{N} \tilde{\varphi}_{N-1}(\lambda_j) = \pm \prod_{j=1}^{N-1} b_j^{a(N-j)}
\]

The spectral measure is related to \( \varphi_N \) and \( \varphi_{N-1} \) by,

\[
\sum_{k=1}^{N} \frac{p_k}{z - \lambda_k} = \frac{\tilde{\varphi}_{N-1}(z)}{\varphi_N(z)}
\]
\[ P_j = \frac{q_{n-1}(\lambda_j)}{q_n'(\lambda_j)} \]

\[
\prod_{j=1}^{N} P_j \prod_{j=1}^{N} q_n'(\lambda_j) = \pm \prod_{j=1}^{N} q_{n-1}(\lambda_j)
= \pm \prod_{j=1}^{N-1} b_j^{2(n-j)}
\]

\[
\prod_{j=1}^{N-1} b_j^{2(n-j)} = \prod_{j=1}^{N} P_j \prod_{i=1}^{N} |\lambda_i - \lambda_j|^2
\]

Proof of (a).

1. Eigenvalues of \( X \) are distinct.

2. \( p_j = \frac{q_{n-1}(\lambda_j)}{q_n'(\lambda_j)} \) cannot be zero because of interlacing of eigenvalues.

And so \( G \) maps \( J_n \) into \( M_n \).
Let $\nu = \sum p_j \delta_{x_j}$ and apply Gram-Schmidt to $1, x, \ldots, x^{n-1}$ (in $L^2(\nu)$) to get $\psi_0, \ldots, \psi_{n-1}$.

Let's expand $x\psi_k(x)$ in this basis,

$$x\psi_k(x) = c_{k,k+1} \psi_{k+1}(x) + \cdots + c_{k,0} \psi_0(x), \quad k \leq n-2$$

$$x\psi_{n-1}(x) = c_{n,n-1} \psi_{n-1}(x) + \cdots + c_{n,0} \psi_0(x)$$

Some observations...

(1) $c_{k,k+1} > 0$

(2) $\langle x\psi_k, \psi_j \rangle = \langle \psi_k, x\psi_j \rangle$

$\Rightarrow c_{k,k+1} = c_{k+1,k}$

$\Rightarrow c_{k,j} = 0$ if $j < k-1$

This matrix of $c$'s (which we will call $H(\nu)$), is a Jacobi matrix. So $H$ maps $M_n$ into $J_n$.

In particular, let $a_k = c_{k,k}$ and $b_k = c_{k-1,k}$. 
All that is left is to show \( G \circ H \) is the identity on \( M_n \) and \( G \) is one-to-one (both of these are just a bit of algebra).