<u>Def.</u> Let $A = (a_{i,j})_{i,j \le N}$ where $a_{i,j}$ are i.i.d $\mathbb{C}N(0,1)$ and let

$$X = \frac{1}{\sqrt{2}}(A + A^*)$$

Then $X \in GUE(N)$ ($\mathcal{H}_N^{(2)}$ in Anderson).

<u>Def.</u> Let $B = (b_{i,j})_{i,j \le N}$ where $b_{i,j}$ are i.i.d $N(0,1)$ and let

$$Y = \frac{1}{\sqrt{2}}(B + B^t)$$

Then $Y \in GOE(N)$ ($\mathcal{H}_N^{(2)}$ in Anderson).

<u>Thm (Joint distribution of eigenvalues).</u> Let $X \in \mathcal{H}_N^{(\beta)}$ be random with law $P_N^{(\beta)}$ for $\beta = 1, 2$. The joint distribution of the eigenvalues $\lambda_1(X) \leq \cdots \leq \lambda_N(X)$ has the following density wrt the Lebesgue measure

$$N!\, \bar{C}_N^{(\beta)} \mathbb{1}_{\{x_1 \leq \cdots \leq x_n\}} |\Delta(x)|^\beta \prod_{i=1}^N \exp(-\beta x_i^2/4)$$

normalization constant

Vandermonde determinant

$$\Delta(x) = \prod_{i<j} (x_j - x_i)$$

A <u>Jacobi matrix</u> is a symmetric (Hermitian) tridiagonal matrix where the off-diagonal terms are strictly positive.

$$X_N = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{N-1} \\ & & b_{N-1} & a_N \end{bmatrix}$$

We want to study the joint distribution of the eigenvalues of X. To do so we will need to do the following,

1. Construct a map from $(a,b) \mapsto (\lambda, ?)$.

2. Compute the Jacobian of the above map.

3. Observe how different densities of (a,b) affect the distribution of λ.

Constructing the map.

(a,b) has $2N-1$ variables, but we only have N eigenvalues. So we must find $N-1$ auxilary variables in order to create the map.

$v_1, \dots, v_N$ orthonormal eigenvectors
$\lambda_1, \dots, \lambda_N$ corresponding eigenvalues

Let $p_k = |\langle v_k, e_1 \rangle|^2$ and they satisfy,

$$p_1 + \cdots + p_N = 1$$

so let $p_1, \dots, p_{N-1}$ be the $N-1$ auxilary variables.

Now we have enough variable to try to find a bijection.
First let's identify (λ, p) with the spectral measure of X_N,

$$\nu_X = \sum_{k=1}^{N} p_k \delta_{\lambda_k}$$

Now we can define the map $G : \mathcal{J}_N \to \mathcal{M}_N$ by

$$G(X) = \nu_X$$

where

$$\mathcal{J}_N = \mathbb{R}^N \times \mathbb{R}^{N-1}_+$$

$$\mathcal{M}_N = \mathbb{R}^N_\uparrow \times \Delta_N$$

<u>Lemma.</u> Let G be defined as above. Then,

(a) G is a bijection

(b) $\prod_{k=1}^{N-1} b_k^{2(N-k)} = \prod_{k=1}^{N} p_k \cdot \prod_{i<j} |\lambda_i - \lambda_j|^2$

(c) The Jacobian determinant of G^{-1} is equal to (up to a sign),

$$J_{G^{-1}}(\lambda, p) = \frac{\prod_{k=1}^{N-1} b_k}{2^{N-1} \prod_{k=1}^{N} p_k}$$

Proving the above lemma will be most of the work, so let's hold off so we can understand some consequences first.

Let (a,b) have joint density $f(a,b)$ wrt the Lebesgue measure on $\mathcal{J}_N$. Then the density of (λ,p) on $\mathcal{M}_N$ (wrt Lebesgue measure) is,

$$g(\lambda,p) = f(a,b)\,\frac{\prod_{k=1}^{N-1} b_k}{2^{N-1}\prod_{k=1}^{N} p_k}$$

What is a 'nice' choice of f?

$$f(a,b) \propto \exp\left\{-\frac{\beta}{4}\Big[\underbrace{\sum_{k=1}^{N} a_k^2 + 2\sum_{k=1}^{N-1} b_k^2}_{\substack{=\mathrm{Tr}(T^2)\\ =\sum_{k=1}^{N}\lambda_N^2}}\Big]\right\}\underbrace{\prod_{k=1}^{N-1} b_k^{\beta(n-k)-1}}_{\text{Cancel parts of above and then use (b) of lemma}}$$

Then we get

$$g(\lambda,p) \propto \exp\left\{-\frac{\beta}{4}\sum \lambda_k^2\right\}\frac{\prod_{k=1}^{N-1} b_k^{\beta(N-k)}}{2^{N-1}\prod_{k=1}^{N} p_k}$$

$$\propto \exp\left\{-\frac{\beta}{4}\sum \lambda_k^2\right\}\prod_{i<j}|\lambda_i-\lambda_j|^{\beta}\prod_{k=1}^{N} p_k^{\frac{\beta}{2}-1}$$

<u>Thm.</u> Let the $a_k \sim N(0,2)$ and $b_k \sim \chi^2_{\beta(n-k)}$ (both independent and independent of each other). Then the eigenvalue of X_N have density proportional to,

$$\exp\left\{-\frac{\beta}{4}\sum_{k=1}^{N}\lambda_k^2\right\}\prod_{i<j}|\lambda_i-\lambda_j|^\beta$$

This density is called the β-Hermite ensemble.

What's next...

1. Show that Gaussian ensembles are β-Hermite ensembles.

2. (If time) Prove the lemma.

Reduction of GOE into Jacobi matrix.

$A \in GOE(N)$

$$A = \begin{bmatrix} a_1 & v_1^t \\ v_1 & B \end{bmatrix}$$

$B \in GOE(N-1)$

Want a $P \in O(N-1)$ such that $Pv = re_1$ and $r = \|v\|$. One choice of P is,

$$P = \mathbb{1} - 2uu^t \quad , \quad u = \frac{v - re_1}{\|v - re_1\|}$$

Householder reflection

$$\hat{P} = \begin{bmatrix} 1 & 0^t \\ 0 & P \end{bmatrix}$$

$$A_1 = \hat{P} A \hat{P}^t = \begin{bmatrix} a_1 & re_1^t \\ re_1 & C \end{bmatrix}$$

Now let's repeat this process for the matrix C.

$$C = \begin{bmatrix} a_2 & v_2^t \\ v_2 & D \end{bmatrix} \qquad \hat{Q} C \hat{Q}^t = \begin{bmatrix} a_2 & r_2 e_1^t \\ r_2 e_1 & Q D Q^t \end{bmatrix}$$

where $r_2 = \|v_2\|$. Repeating this procedure will result in the matrix

$$\begin{bmatrix} a_1 & r_1 & & \\ r_1 & \ddots & \ddots & r_{N-1} \\ & \ddots & \ddots & \\ & r_{N-1} & & a_N \end{bmatrix}$$

Not only is it a Jacobi matrix, but it is β-Hermite.

GOE $\mapsto \beta = 1$
GUE $\mapsto \beta = 2$
GSE $\mapsto \beta = 4$

No corresponding matrices for general β

<u>Proof of (b).</u>

X_k = top-left k-k submatrix of X_N
$\tilde{X}_k$ = bottom-right ... ...

φ_k = characteristic polynomial of X_k
$\tilde{\varphi}_k$ = ... of $\tilde{X}_k$

$\lambda_j^{(k)}$ = eigenvalues of X_k

We have the following recursion,

$$\varphi_k = (z - a_k)\varphi_{k-1} - b_{k-1}^2 \varphi_{k-2}, \quad \varphi_0 = 1, \quad \varphi_{-1} = 0.$$

<u>Fact.</u> Eigenvalues of T_k are strictly interlaced with the eigenvalues of T_{k-1}.

Let $z = \lambda_j^{(k-1)}$ and multiply over all $j \leq k-1$.

$$\prod_{j=1}^{k-1} \varphi_k(\lambda_j^{(k-1)}) = (-1)^{k-1} b_{k-1}^{2(k-1)} \prod_{j=1}^{k-1} \varphi_{k-2}(\lambda_j^{(k-1)})$$

For φ_k and φ_{k-1} we have the relation,

$$\prod_{j=1}^{k} \varphi_{k-1}(\lambda_j^{(k)}) = \pm \prod_{j=1}^{k-1} \varphi_k(\lambda_j^{(k-1)})$$

$$= \pm\, b_{k-1}^{2(k-1)} \prod_{j=1}^{k-1} \varphi_{k-2}(\lambda_j^{(k-1)})$$

Take the product over k and telescope

$k=2$ to $k=N$

$$\prod_{j=1}^{N} \varphi_{N-1}(\lambda_j) = \pm \prod_{j=1}^{N-1} b_j^{2j}$$

We can do the same procedure (in reverse) to the $\tilde{\varphi}_k$'s,

$$\prod_{j=1}^{N} \tilde{\varphi}_{N-1}(\lambda_j) = \pm \prod_{j=1}^{N-1} b_j^{2(N-j)}$$

The spectral measure is related to φ_N and φ_{N-1} by,

$$\sum_{k=1}^{N} \frac{p_k}{z-\lambda_k} = \frac{\tilde{\varphi}_{N-1}(z)}{\varphi_N(z)}$$

$$p_j = \frac{\tilde{\varphi}_{N-1}(\lambda_j)}{\varphi_N'(\lambda_j)}$$

$$\prod_{j=1}^{N} p_j \prod_{j=1}^{N} \varphi_N'(\lambda_j) = \pm \prod_{j=1}^{N} \tilde{\varphi}_{N-1}(\lambda_j)$$
$$= \pm \prod_{j=1}^{N-1} b_j^{2(N-j)}$$

$$\prod_{j=1}^{N-1} b_j^{2(N-j)} = \prod_{j=1}^{N} p_j \prod_{i<j} |\lambda_i - \lambda_j|^2$$

<u>Proof of (a).</u>

(1) Eigenvalues of X are distinct.

(2) $p_j = \tilde{\varphi}_{N-1}(\lambda_j) / \varphi_N'(\lambda_j)$ cannot be zero because of interlacing of eigenvalues.

And so G maps $\mathcal{J}_N$ into $\mathcal{M}_N$.

Let $\nu = \sum_1 p_j \delta_{\lambda_j}$ and apply Gram-Schmidt to $1, x, \ldots, x^{N-1}$ (in $L^2(\nu)$) to get $\psi_0, \ldots, \psi_{N-1}$.

Let's expand $x\psi_k(x)$ in this basis,

$$x\psi_k(x) = c_{k,k+1}\psi_{k+1}(x) + \cdots + c_{k,0}\psi_0(x) \quad , \quad k \leq N-2$$

$$x\psi_{N-1}(x) = c_{N,N-1}\psi_{N-1}(x) + \cdots + c_{N,0}\psi_0(x)$$

Some observations...

(1) $c_{k,k+1} > 0$

(2) $\langle x\psi_k, \psi_j \rangle = \langle \psi_k, x\psi_j \rangle$

$\Rightarrow c_{k,k+1} = c_{k+1,k}$

$\Rightarrow c_{k,j} = 0$ if $j < k-1$

This matrix of c's (which we will call $H(\nu)$), is a Jacobi matrix. So H maps $\mathcal{M}_N$ into $\mathcal{J}_N$.

In particular, let $a_k = c_{k,k}$ and $b_k = c_{k-1,k}$.

All that is left is to show $G \circ H$ is the identity on M_N and G is one-to-one (both of these are just a bit of algebra).