The Semi-Circle Law (Pt 1)

References:
- Anderson / Guionnet / Zeitouni: Section 2.1
- Todd Kemp's Math 247A
- Tao's "Topics in Modern Theory"

Recall from Milind's talk.

\[ M_n = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \]

Symmetric, where
- \( d_i \) i.i.d. \( (0, 1) \)
- \( a_{ij} \) i.i.d. \( (0, 1) \)

Wigner matrix

Why study these? Not sure but maybe because we like i.i.d. & symmetry?

Why the semi-circle law? It's like a central limit theorem.
The eigenvalues of $M_n$ form an "histogram":

$$M_n = \frac{1}{n} \sum_{j=1}^{n} \delta_j(M_n)$$

\[ \text{empirical spectrum distribution (ESD)} \]

M hint said last week:

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**Wigner Semicircle Law:**

\[
\text{random var.} \\
\implies \int f \mu \left( \frac{M_n}{\sqrt{n}} \right) \text{ a.s.} \implies \int f \, d\sigma, \forall f \text{ c.d.f.}
\]

where $\sigma$ is the semicircle law:

\[
\sigma(dx) = \frac{1}{2\pi \sqrt{4-x^2}} \, dx.
\]

This scaling $\frac{1}{\sqrt{n}}$ "makes sense" because $\|M_n\|_2$ concentrates in a region of size $\sim \sqrt{n}$.\]
Most of the work is in showing convergence in probability of the above integrals, so we show conv. in probability instead.

We can make successive weakenings of this statement.

(2) \[ \mathbb{E} \int f \, d\mu_n(M_n/\tilde{\epsilon}_n) \rightarrow \int f \, d\mu. \] Since \( \mathbb{E} \) is a linear functional, we can think of the above as \( \mathbb{E} \mu_n(M_n/\tilde{\epsilon}_n) \) instead:

\[
\int f \, d\tilde{\mu}_n(M_n/\tilde{\epsilon}_n) := \mathbb{E} \int f \, d\mu_n(M_n/\tilde{\epsilon}_n).
\]

Note that if \( f = x^k \), this simplifies.

Let \( M_n = \Lambda^T D \Lambda \) for diagonal \( D \),

we have
\[
\int x^k \mu(C_{\lambda n}(\bar{\mu}_{\lambda n})) = \frac{1}{n} \sum_{j=1}^{n} \left[ \int f_j(C_{\lambda n}(\bar{\mu}_{\lambda n})) \right]^{k} \\
= \frac{1}{n} \sum_{j=1}^{n} D_j^{k} = \frac{1}{n} \text{Tr}(C_{\lambda n}^{k}) \\
= \frac{1}{n} \text{Tr}(\Lambda_{\lambda n}^{T} D_{\lambda n}) = \frac{1}{n} \text{Tr}(C_{\lambda n}^{k}).
\]

**Lemma 1**: \( \int x^k \mu(C_{\lambda n}(\bar{\mu}_{\lambda n})) \to \int x^k \, dx \).

We like this because \[ \int x^k \mu(C_{\lambda n}(\bar{\mu}_{\lambda n})) = \mathbb{E} \left( \frac{1}{n} \text{Tr}(C_{\lambda n}^{k}) \right) \] (Note we assume we can take all moments here.)
Why is this enough?

Issues:

- Using $\overline{\mu}$ instead of $\mu$;
- Using only the moments.

Lemmas:

Show that:

\[
\text{Var}(\langle \mu_n, x^k \rangle) = \text{Var}\left( \frac{1}{n} \text{Tr}(M_n S_n)^k \right) \to 0 \quad \text{V.K.}
\]

Clebyshev: $1 \langle \mu_n, x^k \rangle - \langle \overline{\mu}_n, x^k \rangle$
\[ \frac{p}{30}. \]

* We're only using the moments.

In general, monoglass approx should work (?)

**Lemma 3 (cutoff lemma):**

For $B \subset \mathbb{R}$, for $Q$ a polynomial:

\[
\left< \mu_n, \mathbb{II}(x) Q(x) \right> \to 0 \quad \text{in} \quad n \to \infty \quad \text{as} B.
\]

**Proof of lemma 3:** (Assume lemma 2)

Markov ineq on $\left< \mu_n, 1x \leq B \right>$
\[ P \left( \sum_{x_1} \mathbb{I}_{x_1 \in B} \right) > 3 \] 

\[ \sum_{x_1} \frac{1}{3} \mathbb{I}_{x_1 \in B} \left\{ \mu_n, 1 \times 1^{k-1} \right\} \]

\[ = \frac{1}{3} \left\langle \mu_n, 1 \times 1^{k-1} \right\rangle \cdot \mathbb{I}_{x_1 > B} \]

Let \( u_n = 1 \times 1^{k-1} \mu_n (dx) \), then

\[ \frac{1}{3} \left\langle \mu_n, 1 \times 1^{k-1} \right\rangle \cdot \mathbb{I}_{x_1 > B} \]

\[ = \frac{1}{3} \int_{[-e, e]^{k-2}} \frac{1}{3} d\mu_n = \frac{1}{3} 2 \left( \left\{ x : 1 \times x^{1-1} \right\} \right) \]
\[ e \geq \frac{1}{3} \cdot \frac{1}{\beta^c} \sum_{x=1}^{\infty} d x n \]

Markov, again:

\[ = \frac{1}{3} \beta^c \sum_{k=0}^{\infty} x^{2k} d \mu_k^n \]

\[ = \frac{1}{3} \beta^c \left( \int_0^{\infty} x^{2k} d \sigma + o(\sigma) \right) \]

\[ = C_{1c} := \frac{(2k)!}{(k+1)! \cdot k!} \]

Or by:

\[ 4 \cdot (2k-1) = C_{k-1} \]

\[ C_{1c} \geq 4^{1c} \]
Putting it together:

\[ \text{IPC}(\mathcal{U}^{\mu_n, 1 \times 1, 1 \times 1} > \delta) \]

\[ \leq \frac{4^k}{3 \cdot 3^{\delta c}} \quad \text{wts LHS} \to 0 \quad \text{as } n \to \infty. \]

Take a lim sup on LHS & define

\[ S_k := \limsup_{n \to \infty} \text{LHS}. \]

\[ S_k \leq \frac{4^k}{3 \cdot 3^{\delta c}}. \]

LHS is decreasing in k, RHS is decreasing, so \( S_k \)'s must be zero.
To complete the proof sent-cade hypothesis, use lemmas 1, 2, 3 together with W. Shaggs approximation and some simple inequalities to show \( P(K_{n}, f) - <e, f>_1 > 3 \) \( \to 0 \) \( \forall \epsilon \).

Drew's questions

1. Peter Tao's 6006 notes that \( \tilde{u} \) is subgaussian, so its determinant by 15 moments, and then somehow only lemma 182 are needed (Exercise 2.1.6)?

2. truncation trick: to remove moment assumptions (AGZ Theorem 2.1.21).