Probability Student Seminar Week 2 Sp. 2020

The Semi-Circle Law (Pt 1)

References: • Anderson/Guionnet/Zeitouni Section 2.1
• Todd Kemp's Math 247A
• Tao's "Topics in Matrix Theory"

Recall from Milind's talk,

$$M_n = \begin{bmatrix} d_1 & a_{12} & a_{13} & \cdots \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ & & & d_n \end{bmatrix}$$

symmetric, where $d_i \overset{iid}{\sim} (0, V)$, $a_{ij} \overset{iid}{\sim} (0, 1)$.

Wigner matrix

Why study these? Not sure, but maybe because we like iid & symmetry?

Why the semicircle law? It's like a central limit theorem.

The eigenvalues of M_n form "histogram":

$$\mu_{M_n} := \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j(M_n)} .$$

$\Leftarrow$ empirical spectral distribution (ESD)

Mind snd last week:

Wigner Semicircle Law:

random var.

(1) $$\boxed{\int f \, d\mu_{(M_n/\sqrt{n})}} \xrightarrow{a.s.} \int f \, d\sigma, \quad \forall f \text{ cts bdd.}$$

Where σ is the semicircle law:

$$\sigma(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\, dx .$$

This scaling $\frac{1}{\sqrt{n}}$ "makes sense" because (?)

$\|M_n\|_2$ concentrates in a region of size $\sim \sqrt{n}$.

Most of the work is in showing convergence in probability of the above integrals, so we show conv. in probability instead.

We can make successive weakenings to this statement:

(2) $\mathbb{E}\int f \, d\mu(M_n/\sqrt{n}) \to \int f \, d\sigma$. Since $\mathbb{E}$ is a linear functional, we can think of the measure $\bar{\mu}(M_n/\sqrt{n})$ defined:

$$\int f \, d\bar{\mu}(M_n/\sqrt{n}) := \mathbb{E}\int f \, d\mu(M_n/\sqrt{n}).$$

Note that if $f = x^k$, this simplifies.

Let $\frac{M_n}{\sqrt{n}} = \Lambda^T D \Lambda$ for diagonal D;

we have

$$\int x^k \, d\mu_{(M_n/\sqrt{n})} = \frac{1}{n} \sum_{j=1}^{n} \left[\lambda_j (M_n/\sqrt{n}) \right]^k$$

$$= \frac{1}{n} \sum_{j=1}^{n} D_{jj}^k = \frac{1}{n} \operatorname{Tr}(D^k)$$

by convention $k \pm 1$ power of $M/\sqrt{n}$ ✓

$$= \frac{1}{n} \operatorname{Tr}(\Lambda^T D^k \Lambda) = \frac{1}{n} \operatorname{Tr}(M/\sqrt{n})^k.$$

$$\boxed{\underline{\text{Lemma 1}}: \int x^k \, d\bar{\mu}(M_n/\sqrt{n}) \to \int x^k \, d\sigma}$$

We like this because

$$\int x^k \, d\bar{\mu}(M_n/\sqrt{n}) = \mathbb{E} \frac{1}{n} \operatorname{Tr}(M_n/\sqrt{n})^k$$

(Note we assume we can take all moments here.)

Why is this enough?

Issues!

- Using $\bar{\mu}$ instead of μ;
- Using only the moments.

- $\bar{\mu}$ instead of μ:

define:

$$\mu_n := \mu_{M_n/\sqrt{n}}$$

$$\bar{\mu}_n := \bar{\mu}_{M_n/\sqrt{n}}$$

$$\langle \mu, f \rangle := \int f \, d\mu.$$

Lemma 2 Show that

$$\mathrm{Var}(\langle \mu_n, x^k \rangle) = \mathrm{Var}\left(\frac{1}{n} \mathrm{Tr}(M_n/\sqrt{n})^k\right) \to 0 \quad \forall k.$$

Chebyshev: $|\langle \mu_n, x^k \rangle - \langle \bar{\mu}_n, x^k \rangle|$

$\xrightarrow{P} 0$.

☆ We're only using the moments.

In principle, Weierstrass approx should work(?)

Lemma 3 (cutoff lemma):

For $B > 4$, for Q a polynomial:

$$\langle \mu_n, \mathbb{1}_{|x|>B}(x)\, Q(x) \rangle \xrightarrow[n\to\infty]{} 0 \text{ in prob.}$$

Proof of lemma 3: (assume lemma 1)

Markov ineq on $\langle \mu_n, |x|^k \mathbb{1}_{|x|>B} \rangle$

gives

$$\mathbb{P}\left(\langle \mu_n, |x|^k \mathbb{1}_{|x|>B} \rangle > \varepsilon\right)$$

$$\leq \frac{1}{\varepsilon} \mathbb{E} \langle \mu_n, |x|^k \mathbb{1}_{|x|>B} \rangle$$

$$= \frac{1}{\varepsilon} \langle \bar{\mu}_n, |x|^k \mathbb{1}_{|x|>B} \rangle.$$

Let $\nu_n = |x|^k \bar{\mu}_n(dx)$, then

$$\frac{1}{\varepsilon} \langle \bar{\mu}_n, |x|^k \mathbb{1}_{|x|>B} \rangle$$

$$= \frac{1}{\varepsilon} \int_{[-B,B]^c} 1 \cdot d\nu_n = \frac{1}{\varepsilon} \nu(\{x : |x|^k > B^k\}).$$

$$= \frac{1}{\varepsilon} \cdot \frac{1}{\beta^k} \int_0^\infty |x|^k \, d\nu_n$$

Markov, again

$$= \frac{1}{\varepsilon \beta^k} \int_0^\infty x^{2k} \, d\tilde{\mu}_n$$

$$= \frac{1}{\varepsilon \beta^k} \left(\underbrace{\int_0^\infty x^{2k} \, d\sigma}_{= C_k := \frac{(2k)!}{(k+1)!\,k!}} + o(1) \right).$$

or by $\frac{4 \cdot (2k-1)}{2k+2} = C_{k-1}$

$$C_k \asymp 4^k,$$

putting it together:

$$\mathbb{P}\left(\langle \mu_n, |x|^k \mathbb{1}_{|x| > \beta} \rangle > \varepsilon\right)$$

$$\leq \frac{4^k}{\varepsilon \cdot \beta^k}.$$

WTS LHS $\to 0$

as $n \to \infty$.

Take a lim sup on LHS & define

$$S_k := \limsup_{n \to \infty} (\text{LHS}).$$

$$S_k \leq \frac{4^k}{\varepsilon \cdot \beta^k}.$$

LHS is increasing in k, RHS is decreasing, so S_k's must be zero.

$\blacksquare$

To complete the proof of semi-circle law, use lemmas 1, 2, 3 together + Weierstrass approximation & some simple inequalities, to show

$$\mathbb{P}(|\langle \mu_n, f\rangle - \langle \sigma, f\rangle| > \varepsilon) \to 0 \quad \forall \varepsilon.$$

Drew's questions

- Terence Tao's book notes that $\bar{\mu}_n$ is subgaussian, so its determined by its moments, and then somehow only lemmas 1 & 2 are needed (Exercise 2.4.6)?

- Truncation trick: to remove moment assumptions (AGZ Theorem 2.1.21).