Qualifying Exam Syllabus

Milind Hegde

Committee: Craig Evans (Chair), Alan Hammond (Advisor),
Jim Pitman, Alistair Sinclair (EECS).

1 Major topic: Probability Theory (Probability)

References: Durrett chapters 1-3, 5-6, 8.

- **Measure theoretic preliminaries:** σ-algebras, Carathéodary extension, π-λ systems, random variables, probability measures, expectation, dominated convergence theorem, Fubini's theorem.
- **Processes, Distributions, and Independence:** Independence of σ-algebras, Kolmogorov Zero-One Law, Hewitt-Savage Law, Exchangeability, Borel-Cantelli, Kolmogorov Extension.
- **Random Sequences:** Modes of convergence, uniform integrability, tightness, Weak and Strong Laws of Large Numbers, Glivenko-Cantelli, Portmanteau theorem, Continuous mapping theorem, Skhorohod representation.
- **Characteristic Functions:** Uniqueness and Lévy's Continuity Theorem, Poisson Convergence, Central Limit Theorem.
- **Conditioning and Disintegration:** Conditional expectation and probability, regular conditional probabilities, disintegration, conditional independence.
- **Martingales:** Filtrations and stopping times, martingale property, optional stopping and sampling, maximum and upcrossing inequalities, martingale convergence, uniform integrability.
- **Markov Chains:** Countable state space Markov chains, recurrence and transience, stationary measures, asymptotic behavior.
- **Brownian Motion:** Existence and path properties, Strong Markov Property, Blumenthal Zero-One Law, hitting times and reflection principle.

2 Major topic: Partial Differential Equations (Analysis)

References: Evans chapters 2-3, 5-6, 8.

- **Functional Analysis Preliminaries:** Hahn-Banach Theorem, Baire Category Theorem, Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Principle, Banach-Alaoglu Theorem, Compact Operators.

- **Laplace's Equation:** Fundamental solution, mean-value formula, properties of harmonic functions, Green's function, energy methods. (Evans: 2.2)
- **Heat Equation:** Fundamental Solution, mean-value formula, properties of solutions, energy methods. (Evans: 2.3)
- **Nonlinear First-Order PDE:** Method of characteristics, Hamilton's ODE, Legendre Transform, Hopf-Lax formula, scalar conservation laws, shocks, entropy condition, Lax-Oleinik formula, Riemann's Problem. (Evans: 3.2-3.4)
- **Sobolev Spaces:** Hölder & Sobolev spaces, weak derivatives, approximation, extensions, traces, Sobolev Inequalities, compactness, Poincaré's Inequalities, difference quotients, differentiability a.e. (Evans: 5.1-5.8)
- **Second-Order Elliptic Equations:** Existence of weak solutions, regularity, maximum principles, eigenvalues and eigenfunctions of symmetric elliptic operators. (Evans: 6.1-6.5.1)
- **Calculus of Variations:** First variation, Euler-Lagrange Equation, second variation, existence of minimizers, regularity, constraints. (Evans: 8.1-8.4)
- **Hamilton-Jacobi Equations:** Viscosity solutions, uniqueness, control theory, dynamic programming.

3 Minor topic: KPZ (Probability)

References: Corwin 2011 survey, Romik chapters 1-2, Corwin-Hammond 2014 paper.

- **KPZ Equation:** KPZ Equation, ill-posedness, Hopf-Cole solution via stochastic heat equation, characteristic scaling exponents, conjectured universality. (Corwin 2011 survey)
- **Longest increasing subsequence problem:** Robinson-Schensted-Knuth algorithm, Young tableaux, Plancherel measure, Fredholm determinants, Poissonian Last Passage Percolation, Tracy-Widom distribution. (Romik: chapters 1-2)
- **Airy Line Ensemble:** Brownian Gibbs property, Airy process, Brownian LPP, Karlin-McGregor formula for non-intersection. (Corwin-Hammond 2014 paper)

References:

I. Corwin, *The Kardar-Parisi-Zhang Equation and Universality Class*, 2011.
I. Corwin, A. Hammond, *Brownian Gibbs Property for Airy Line Ensembles*, 2014.
R. Durrett, *Probability: Theory and Examples*.
C. Evans, *Partial Differential Equations*.
D. Romik, *The Surprising Mathematics of Longest Increasing Subsequences*.

4 Transcript

The following is an approximate record of my qualifying examination, as best I remember it. First I was asked what I'd like to start with, and I picked PDE.

Craig: Ok, write down the heat equation in $\mathbb{R}^n$ with initial condition g at $t = 0$.

I did this.

Craig: Now do you know a formula for the solution to this?

I write $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy$, mentioning that Φ is the fundamental solution. On being asked to define the fundamental solution, I start writing down the fundamental solution for Laplace's equation...

Jim: That looks Laplacian.

I realise my mistake, write down the correct fundamental solution but with an integral! Craig and Jim make some noises, maybe point out that what I've got is a number? At this point I've realised the second mistake and am in the process of erasing the integral and just writing $(4\pi t)^{-n/2}e^{-|x|^2/4t}$.

Craig: Ok, so you've got a solution. What can you tell me about uniqueness?

I say that the solution is unique given a growth condition; Craig prompts me for the condition, and I continue by saying the solution has to be assumed to be bounded by $Ae^{a|x|^2}$ for some a, A and all $x, t \in \mathbb{R}^n \times (0, \infty)$.

Craig: Tell me about what it means to say the heat equation displays infinite propagation speed.

I say that if $g \geq 0$ and is actually positive at some point, then by the integral form of the solution it follows that $u(x,t) > 0$ at any $x \in \mathbb{R}^n$ for any $t > 0$.

Craig: What kind of regularity do you need to assume on g? What if it's identically zero and is 1 at $x = 0$?

Milind: Right, I suppose it would be sufficient if g is continuous.

Craig: Alright, let's move on. Have you studied viscosity solutions?

Milind: Yes.

Craig: Ok, write down the definition of a viscosity solution.

I wrote down the Hamilton-Jacobi PDE and defined what a viscosity solution is; I wrote on the board the case where $u - v$ has a local maximum at (x_0, t_0) implies v satisfies an inequality version of the Hamilton-Jacobi PDE and said orally that if $u - v$ has a local minimum the inequality reverses, but Craig made me modify the statement on the board with parantheses to cover both cases.

Craig: Tell me about dynamic programming and the connection with viscosity solutions.

I wrote down the ODE system involved in dynamic programming, mentioned the assumption that f be Lipschitz, and defined the cost associated to a solution path using the running cost

r and terminal cost g. I then wrote down the value function u and said that the connection to viscosity solutions is that u satisfies a Hamilton-Jacobi PDE except with a terminal condition instead of an initial one.

Craig: Can you write down the PDE that u satisfies?

Milind: Uh, I don't quite remember it exactly but let me try to figure it out.

Craig: Neither do I, try to give me an approximate form of it.

I write down $u_t + \min_a r(x,a) \cdot Du$...

Craig: What you've got doesn't make sense — r is a scalar but Du is a vector, and you're taking their dot product.

Milind: Right...

Craig: Look at what you've written on the board. What quantity there is a vector that might fit where you've got r?

Milind: Ok, I don't remember the PDE exactly but I know u satisfies an "infinitesimal" formula and I can get the PDE from that.

I write down

$$u(x,t) = \inf_{\boldsymbol{\alpha} \in \mathcal{A}} \left\{ \int_t^{t+h} r(\boldsymbol{x}(s), \boldsymbol{\alpha}(s))\, ds + u(\boldsymbol{x}(t+h), t+h) \right\}.$$

Milind: So heuristically ...

$$\frac{u(\boldsymbol{x}(t+h), t+h) - u(x,t)}{h} = \inf_{\boldsymbol{\alpha} \in \mathcal{A}} \left\{ \frac{1}{h} \int_t^{t+h} r(\boldsymbol{x}(s), \boldsymbol{\alpha}(s))\, ds \right\}.$$

Craig: On the right hand side you've got an infimum so you can't move $u(\boldsymbol{x}(t+h), t+h)$ to the other side. What you should be doing is...

Milind: Right, I should take $u(x,t)$ to the right side. Then I would get (I write it down correctly with everything inside the inf and take $h \to 0$.)

$$u_t + \min_a \{\boldsymbol{f}(x,a) \cdot Du(x,t) + r(x,a)\} = 0.$$

Craig: So what is the connection of this to the value function?

Milind: The value function is the viscosity solution to this PDE.

Craig: Ok, and what is the important fact you're invoking?

I'm confused by what he means, but then offer that the viscosity solution is unique.

Craig: Yes, exactly. So if you have some numerical scheme to compute the solution to the PDE, by uniqueness you know that it must be the value function. Ok, so in class just now I joked about how I was going to ask you about constraints here after just teaching it, but I'm too tired to complete the joke.

Jim: Oh, to see if he can remember something he learned half an hour ago?

Craig: Well, I haven't even taught viscosity solutions and he seems to remember it. (says something about remembering the future). Anyway, I'll pass you on PDE, you're fine.

Jim: So what would you like to do next, probability or KPZ?

Milind: Probability, I think.

Jim: Yes, I guess that would make more sense. Ok, write what I say on the board: "Suppose ϕ is the characteristic function of X." What does this mean?

Milind: It means that $\phi(t) = \mathbb{E}[e^{iXt}]$ for all t.

Jim: Now write "ϕ_n are the characteristic function of X_n and $\phi_n(t) \to \phi(t)$ for all t." Can you say much about ϕ?

Milind: No, not much.

Jim: Not much, right?

Milind: Well, you can say $\phi(0) = 1$, but that's not very helpful.

Jim: (laughs) Yes, I suppose you can. Now can you elaborate on what's on the board?

Milind: Well, if you know beforehand that ϕ is a characteristic function itself, of some random variable X, then this means that $X_n \to X$ in distribution. If you don't know that ϕ is a characteristic function but you know that the X_n are tight, then you can find a subsequence such that X_{n_k} converges in distribution to some X, and ϕ will be the characteristic function of X. Or if you don't know this either but you know that ϕ is continuous at 0, then it turns out that X_n are tight, so like before some subsequence converges to some X whose characteristic function is ϕ.

Jim: Good. Ok, so go over to the side of the board and write down the definition of tightness.

I go over and write the definition in terms of ε; Jim seemed to have been hoping for the equivalent definition with $\limsup$, but he's fine with this too.

Jim: Can you outline the proof of how continuity at zero gives tightness?

Milind: It's the general principle that behaviour of the characteristic function at 0 gives information about the decay at infinity. So here you can get a formula of the mass assigned by the distributions outside an interval in terms of ϕ around 0 which is uniform in n. So it gives tightness.

Jim: Yes, I don't remember the exact formula off the top of my head either. Does anybody else want to ask him about characteristic functions? He seems to thoroughly understand this so let's move onto uniform integrability. Go over to the side of the board and write down the definition.

Milind: It's a very similar definition to tightness. (I write it down.)

Jim: Ok, write down this on the board: "Suppose X_n is a martingale with respect to $(\mathcal{F}_n)$." What can you say about a martingale which is uniformly integrable?

Milind: Well, it's a general fact not relying on the martingale property that if X_n are u.i. and converge a.s., then the convergence also happens in L^1.

Jim: Yes, but I want a characterisation for martingales. Write them down as bullet points.

Milind: Ok, one fact is that a martingale X_n is u.i. if and only if it's a Doob martingale, meaning $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ for some X which is L^1. I guess being L^p bounded for some $p > 1$ is sufficient for u.i., but again this isn't special to martingales.
(I don't remember if I also said that a u.i. martingale will converge a.s. and in L^1.)

Jim: Yeah, don't write down the second one. Let's focus on the first one. Write down this on the board: "$\mathbb{E}[X_T]$". (I write it down with a lower case t). No, I want the T to be capital, I want to emphasise that it's going to be a random time. What can you say about this?

Milind: Well, under certain conditions on T, like for example that it's bounded a.s., it's true that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$. This is the optional stopping theorem.

Jim: Careful, you need an extra adjective on T.

Milind: It just has to be a stopping time. (I hadn't realised that Jim had just called T a random time till now).

Jim: Yes, exactly. So what happens on the event $\{T = \infty\}$?

Milind: Hmm. I think this formula should be true...

$$\mathbb{E}[X_T \mathbb{1}_{T<\infty}] = \mathbb{E}[X_0]$$

Jim: No, that's not true... do you have a candidate for what X_∞ might be?

Milind: Right, since X_n are u.i., X_n is a Doob martingale with X. And that can be considered as X_∞. Then this formula should be true

$$\mathbb{E}[X_T] = \mathbb{E}[X_0 \mathbb{1}_{T<\infty}] + \mathbb{E}[X_\infty \mathbb{1}_{T=\infty}].$$

Jim: Hmm, is that true? I'm not sure, actually...

Milind: Actually, this is true

$$\mathbb{E}[X_T] = \mathbb{E}[X_\infty].$$

Jim: Yes! How might you prove that?

Milind: Well, let me try breaking it up based on the value of T. Then,

$$\mathbb{E}[X_T] = \sum_{k=0}^{\infty} \mathbb{E}[X_k \mathbb{1}_{T=k}] + \mathbb{E}[X_\infty \mathbb{1}_{T=\infty}].$$

Jim: No, I actually don't think that's going to work. You're trying to break up T based on its value, but it's a random object — can you treat it as a whole?

Milind: (Based on the somewhat cryptic hint I got an idea.) I think this formula should be true, which would imply what I'm trying to show:

$$X_T = \mathbb{E}[X_\infty \mid \mathcal{F}_T].$$

This is true just because X_n are uniformly integrable.

Jim: Yes, good. Can you define what F_T is?

I define it.

Jim: Ok, that's good. Does anybody else want to ask Milind anything about martingales? Alistair?

Alistair: Yes, can you give me some sufficient conditions under which optional stopping holds?

Milind: It's sufficient if T is a.s. bounded by some $K < \infty$. Otherwise, you basically want to have a condition which allows you to apply dominated convergence to $X_{T\wedge n}$. So it would be enough if $X_{T\wedge n}$ were bounded, and I think also if $\mathbb{E}[T] < \infty$.

Jim: No, that's not true... you're probably thinking about Brownian motion, where $\mathbb{E}[T] < \infty$ is enough, which is Wald's lemma. But it's not true for general martingales — there's one counterexample for martingales which you should always remember. What is it?

(I don't know what example Jim is talking about, and am thinking about it when Alistair brings the discussion back. Jim was talking about the gambler's double or nothing martingale, as he told me afterwards.)

Alistair: Can you apply optional stopping to get a formula for the probability that the simple symmetric random walk on $\mathbb{Z}$, started at 0, hits some $-a$ before b?

Milind: Yes. If X_n is the random walk, X_n is a martingale, and let $T = \min\{n : X_n = -a \text{ or } b\}$. Then T is a.s. finite, so $X_T = -a$ or b with probability 1. If $p = \mathbb{P}(X_t = -a)$, we can apply optional stopping to get

$$0 = \mathbb{E}[X_0] = \mathbb{E}[X_T] = p(-a) + (1-p)b \implies p = \frac{b}{a+b}.$$

And we can apply optional stopping because X_T is bounded between $-a$ and b. (I should have said $X_{T\wedge n}$ is bounded by them.)

Alistair: Ok. Can you get a bound on $\mathbb{E}[T]$?

Milind: Yes. $X_n^2 - n$ is also a martingale, so...

Alistair: What are you worried about?

Milind: I'm thinking about how I can justify optional stopping. Anyway, assuming optional sampling is applicable, I'll get

$$\mathbb{E}[T] = \mathbb{E}[X_T^2] = pa^2 + (1-p)b^2 = \frac{ba^2}{a+b} + \frac{ab^2}{a+b} = ab.$$

Alan: I think you've made an arithmetic mistake. There should be an $a^2 + b^2$...?

Jim: No, he's fine. You take an ab out and have an $(a+b)$ left inside, which cancels.

Milind: Ok. And to justify optional stopping, T is bounded by a geometric random variable with probability of going up $(a+b)$ times consecutively. This has finite expectation, so...

Jim: Yes, that gives domination by an integrable random variable. Are you satisfied, Alistair? Ok, does anybody have any more questions in probability?

Alan: Suppose A is a Lebesgue measurable subset of the positive real line with positive measure. Given some $\varepsilon > 0$, what can you tell me about the existence of an open set O such that $m(A \cap O)/m(O) > 1 - \varepsilon$?

Milind: I think I can take O to be an interval I. By the definition of the Lebesgue measure

$$m(A) = \inf \left\{ \sum_{k=1}^{N} m(I_k) \mid A \subseteq \bigcup_{k=1}^{N} I_k, N \in \mathbb{N} \right\}.$$

(I'm confused about whether I need to take a countable number of intervals or if finite is ok, and switch between the two a couple times, but settle on finite.)

Milind: (I attempt to draw a diagram of the situation but realise I don't know how to represent an arbitrary measurable set that isn't an interval. Jim laughs when I make this comment.)

Milind: Ok, so now given ε, there has to exist a collection of intervals so that

$$m(A) \geq (1 - \varepsilon) \sum_{k=1}^{N} m(I_k.)$$

(I'm still wondering whether finite is ok. In fact as I realised the next day, you need a countable collection.)

Alan: It looks like you're worried about whether a finite collection suffices. That might be at the heart of the problem. [or something like that]

Milind: Certainly it won't hurt if I take a countable union, so let's go with that. [The committee chuckles, and I think for a little bit more.] So now since the intervals cover A, I can write

$$\sum_{k=1}^{\infty} m(A \cap I_k) = m(A) \geq (1 - \varepsilon) \sum_{k=1}^{\infty} m(I_k).$$

So there has to be at least one interval I satisfying $m(A \cap I) \geq (1 - \varepsilon) m(I)$.

Alan: Yes, ok. Now suppose I give you some additional structure for A, namely A has the property that if $a, b \in A$, then $a + b \in A$. What structural statement about A can you make now?

Milind: I think that A must contain an interval then. I think this is the Steinhauss lemma...

Alan: Try thinking about what has to happen regarding containment for large x.

Milind: Well, if A does contain an interval then the additive property means it contains all sufficiently big x, so A would contain a half-ray.

Alan: Your statement about containing an interval will eventually be correct, but you don't know that yet. So how can you go about proving A contains all large x?

(I'm stumped by this and am also still thinking about how to prove that A contains an interval, for maybe a minute or two.)

Alan: Ok, let me give a little more guidance. What I asked you earlier about a dense interval is not completely irrelevant to this problem.

Milind: Right...

Alan: Think about what you can say about the membership in A of twice the midpoint of the interval I from earlier.

Milind: Ok, so if $I = (a, b)$, you're asking if $a + b \in A$. This should be true. Because if I can find any x such that $\frac{a+b}{2} - x$ and $\frac{a+b}{2} + x$ are both in A, then $a + b \in A$ too.

Alan: You might need some condition on x, but yes.

Milind: Right, probably $\frac{a+b}{2} \pm x$ should lie in I or something. So why does such an x have to be there...

Alan: You also have a density assumption on I.

Milind: Ah ok! So I know that only an ε measure of x can be such that $\frac{a+b}{2} + x$ fail to be in A, and the same for $\frac{a+b}{2} - x$. But since ε is small and represents the proportion of I's measure, I have more x's, so at least one has to work. So $a + b \in A$.

Alan: Ok. Was there anything specific about this argument to $a + b$?

Milind: No. I didn't need to take the same x on both sides of $\frac{a+b}{2}$, so the argument actually gives me a small interval around $a + b$ which lies in A. So then like we said before A has to contain an infinite half-ray.

Craig: Ok, would you like a break now?

Milind: I'm alright, whatever everyone prefers.

Craig: Let's take a break.

Jim: Go drink some water.

Alan: I'm not asking you this, but at this point the next natural question would be to consider a measurable function which has a similar additive property.

Craig: Yes, but...

Alan: I'm not asking this.

(We take a break, and I'm waiting outside when Jim sees me and says they're waiting for me.)

Jim: Ok, we're onto KPZ now. Alan?

Alan: I find it a little strange to be asking these things since Milind and I talk about these things every week. Anyway, I don't see this on your syllabus, but why don't you give a five minute talk on the Karlin-McGregor formula.

Milind: Ok. Actually, it is on my syllabus.

(I give a short talk, presenting the set up, the theorem, what it means in the case when we're considering a Markov process on $\mathbb{R}$, and give an outline of the proof. Jim is impressed by the proof and comments "very clever" and also points out that I needed the process to have the strong Markov property.)

Alan: Alright, now why don't you give a short speech on the KPZ equation, it's ill-posedness and notion of solution, the characteristic scaling and what we mean by universality.

Milind: (I talk about what he asked. At one point Craig interrupts and asks what I mean by a narrow-wedge solution which I'd mentioned. I make a few mistakes with scaling (negative exponent vs. positive etc.) and Alan corrects them, doesn't seem to mind.)

Jim: Ok, I think we're done. Everybody satisfied?

(They send me out of the room and about a minute later bring me back in to congratulate me and are signing some papers.)