A Summary

of

The Complete Intersection Theorem

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The complete intersection theorem gives the answer to the question: What is the maximum size of a family of \( k \)-subsets of \([n] = \{1, \ldots, n\}\) such that every pair of sets from the family has intersection size at least \( t \)?

This is an extension of the Erdős-Ko-Rado Theorem, which says that in the case of \( t = 1 \) and \( n \geq 2k \), the answer is \( \binom{n-1}{k-1} \).

This theorem and the next show that up to permutations, the maximum family in all cases (except if \( n = 2k, t = 1 \)) has the property that the required \( t \)-intersection occurs in the first \( t + 2r \) elements, for a particular \( r \) defined later. This type of family of sets is denoted

\[
\mathcal{F}(i) = \left\{ F \in \binom{[n]}{k} : |F \cap [t+2i]| \geq t+i \right\}
\]

Also, the set of all \( t \)-intersecting families of \( k \) subsets is denoted \( \mathcal{I}(n,k,t) \).

An important idea that is central to the proof is that of left compression, which is described next.

Left Compression

The left shifting operator \( L_{ij} \), where \( i < j \), which acts on sets of the family \( A \) is defined as

\[
L_{ij}(A) = \begin{cases} 
A' = A \setminus \{j\} \cup \{i\} & \text{if } i \notin A, j \in A, A' \notin A \\
A & \text{otherwise}
\end{cases}
\]

In terms of binary vectors, if the vector corresponding to \( A \) has a 1 after a 0, then the vector with the two positions swapped is the vector corresponding to \( L_{ij}(A) \) (if it is not already in \( A \)).

Also define \( L_{ij}(A) \) as the family obtained by left-shifting the sets in \( A \), i.e.

\[
L_{ij}(A) = \{ L_{ij}(A) : A \in A \}
\]

Then we say that a family of sets is left-compressed if for all \( i < j \leq n \),

\[
L_{ij}(A) = A
\]

We call \( L\mathcal{I}(n,k,t) \) the set of families in \( \mathcal{I}(n,k,t) \) which are left-compressed.

Note that the \( t \)-intersecting property of a \( t \)-intersecting family is maintained on left-shifting it. This is important, as it allows the search for maximum families to be restricted to only those which are left-compressed, i.e. to \( L\mathcal{I}(n,k,t) \), and is proved next.
Lemma 1. If $A \in I(n,k,t)$, then $L_{ij}(A) \in I(n,k,t)$, and $|A| = |L_{ij}(A)|$.

Proof. Let $A_1, A_2 \in A$. Since after action by the left shifting operator, the only potential positions of change are $i$ and $j$, it is enough to look at the change in intersection of the two sets at only these two positions.

There are three cases: Under the action of $L_{ij}$, either both $A_1$ and $A_2$ are modified, neither are modified, or exactly one is modified.

1. If both are modified, the intersection is maintained, as this means $i \notin A_1, A_2$ and $j \in A_1, A_2$, while $i \in L_{ij}(A_1), L_{ij}(A_2)$ and $j \notin L_{ij}(A_1), L_{ij}(A_2)$. This means that $i$ has replaced $j$ in the intersection, maintaining the size of the intersection.

2. If neither are modified, clearly the intersection is maintained.

3. There are three sub-cases if only one of the two sets (say $A_1$) is modified, depending on the reason for $A_2$ not being modified. As $A_1$ is modified, we have that $i \notin A_1$ and $j \in A_1$. Here $!$ is a wildcard, and represents a 1 or 0.

   a) $i \in A_2$:

      Here $i \in L_{ij}(A_1) \cap L_{ij}(A_2)$, while $j \in A_1 \cap A_2$ depends on whether $j \in A_2$. So the intersection size either remains the same or increases, but does not decrease.

   b) $j \notin A_2$:

      Here again, depending on whether $i \in A_2$ or not, the intersection size after shifting may remain the same or increase by 1, but will not decrease.

   c) $A_3 := A_2 \setminus \{j\} \cup \{i\} \in A$.

Since $A$ is $t$-intersecting, $|A_1 \cap A_3| \geq t$. Since $i, j \notin A_1 \cap A_3$, we have

$$|A_1 \cap A_2| = |A_1 \cap (A_3 \setminus \{i\} \cup \{j\})| = |A_1 \cap A_3| \geq t$$

It is clear that the cardinality of $A$ is unaffected by left-shifting it. ■
Theorem. Let $M(n,k,t)$ denote the maximum possible size of the family described above.

1. If $n = 2k$, $t = 1$,
   $$M(n,k,1) = M(2k,k,1) = \binom{2k-1}{k-1}$$
2. If $1 \leq t \leq k \leq n$ and $n > 2k - t$, and $r$ is the largest integer such that
   $$(k - t + 1) \left(2 + \frac{t-1}{r+1}\right) < n < (k - t + 1) \left(2 + \frac{t-1}{r}\right)$$
   (where if $r = 0$, the RHS is taken to be $\infty$)
   $$M(n,k,t) = |\mathcal{F}(r)|$$
3. If $t > 1$ and in case 2 strict inequality is not possible, i.e. for some $r$,
   $$n = (k - t + 1) \left(2 + \frac{t-1}{r+1}\right)$$
   then
   $$M(n,k,t) = |\mathcal{F}(r)| = |\mathcal{F}(r+1)|$$

The main idea of the proof is to show that any maximum sized $t$-intersecting family will have a characterising property of $\mathcal{F}(r)$, namely that it will be invariant under permutations of the first $t + 2r$ elements (when the sets are represented as $n$-length binary vectors.)

The proof of this goes by contradiction: in case a maximum family is only invariant under permutations till $\ell < t + 2r$, a new family is constructed, which is larger than the original and has the property that it is invariant under permutations up till $\ell + 1$.

This result and its proof is presented in the next lemma:

Lemma 2. Let $\mathcal{A} \in \mathcal{L}(n,k,t)$ be a maximum $t$-intersecting family with $n > 2k - t$, $t \geq 2$, and, for some non-negative integer $r$, 
   $$n < (k - t + 1) \left(2 + \frac{t-1}{r}\right)$$
Then $\mathcal{A}_{i,j} = \mathcal{A}$ for all $i, j \leq t + 2r$.

Some notation: for a set $A$, by $A_{i,j}$ we denote the set obtained by swapping the $i$ and $j$ coordinates of $A$ (looking at $A$ as a binary $n$-length vector). Then by $\mathcal{A}_{i,j}$ we denote the family of sets obtained by swapping the $i$ and $j$ coordinates in each of the sets in $\mathcal{A}$.

Thus the theorem says that a maximum family has the property that it is invariant under any permutation of the first $t + 2r$ elements.

The strange looking upper-bound on $n$ from which $r$ is defined will arise as a sufficient condition for a required inequality to hold during the course of the proof.
Proof. The idea of the proof is as follows: We denote by $\ell$ the maximum position up to which $A$ is invariant under permutation. So there must be some sets that no longer lie in $A$ when the $i$th coordinate (for some $i \leq \ell$) is transposed with the $(\ell + 1)$th coordinate. We collect these sets in $A'$. Then we show that if $\ell < t + 2r$, we can use the non-empty $A'$ to construct a larger $t$–intersecting family, which will be a contradiction.

Suppose the lemma is false, i.e. $\ell < t + 2r$, where $A_{i,j} = A$ for $i, j \leq \ell$. Define $A'$ as

$$A' = \{ A \in A : A_{i,\ell+1} \notin A \text{ for some } i \leq \ell \}$$

Now we partition $A'$ based on the number of elements contained in $[1, \ell]$:

$$A'(i) = \{ A \in A' : |A \cap [1, \ell]| = i \}$$

$$A' = \bigcup_{i=1}^{\ell} A'(i)$$

Since we have assumed the lemma is false, we have equivalently that $A' \neq \emptyset$. We will now show that each of $A'(i)$ is empty, so that $A' = \emptyset$, making the lemma true.

First we will show that $\ell + 1 \notin A$ if $A \in A'$, which will be used frequently implicitly now onwards.

**Lemma 3.** If $A \in A'$, then $\ell + 1 \notin A$.

Proof. It is easy to prove the contrapositive. If $\ell + 1 \in A$, then $A_{i,\ell+1} = L_{i,\ell+1}A \in A$ for all $i \leq \ell$, as $A$ is left compressed. So by the definition of $A'$, $A \notin A'$.

The idea used in this lemma is that left-shifting, in this case, had precisely the same effect as transposition. As $A$ is left-compressed, this connection can be used easily to talk about transpositions as well.

We also need to count the size of $A'(i)$. To do this, we make an observation similar to the connection between left-shifting and transposing two positions, namely that transposing two positions is the same as transposing the first with a tertiary position, and then transposing the tertiary position with the originally second position. (Similar to the standard way of swapping two variables in computer programming.)

**Lemma 4.** If $A \in A'$, then $A^* \in A'$, where $A^*$ is any set obtained by applying any permutation to the first $\ell$ positions of $A$.

In other words, if $A = B \cup C$, where $B = A \cap [\ell]$, $C = A \cap [\ell + 1, n]$, then $B' \cup C \in A'$, for every $B' \subset [\ell]$ with $|B'| = |B|$.

Proof. We show that transposing any two positions of $A$ will give a set which is still in $A'$. Since any permutation can be expressed as a sequence of transpositions, this will prove the lemma. So it has to be shown that if $A \in A'$, then $A_{i,j} \in A'$ for $i, j \leq \ell$.

Let $i$ be such that $A_{i,\ell+1} \notin A$ and $j \leq \ell$. Then, since from the previous lemma $\ell + 1 \notin A$, $i \in A$ is true.

Now $(A_{i,j})_{i,\ell+1} = A_{i,\ell+1} \notin A$. So we have that $A_{i,j} \in A'$.

The argument is clearer diagrammatically:
Now consider an arbitrary transposition $A_{i,j}$. If $i, j \not\in A$, then $A_{i,j} = A \in A'$. If at least one of $i, j$ belongs to $A$, the above argument will suffice, provided that $i \in A \in A' \implies A_{i,t+1} \not\in A$. This is proved in the next lemma.

**Lemma 5.** If $A \in A'$ and $i \in A$ with $i \leq t$, then $A_{i,t+1} \not\in A$.

**Proof.** Suppose the opposite is true, i.e. there is a set $A \in A'$ with $i \in A$ such that $A_{i,t+1} \in A$.

Consider $j \leq t$. If $j \in A$, $A_{j,t+1} = (A_{i,t+1})_{i,j} \in A$, since all sets in $A$ are invariant under permutations up to $t$. If $j \not\in A$, since $t+1 \not\in A$, $A_{j,t+1} = A \in A$.

Hence $A_{j,t+1} \in A$ for all $j \leq t$, implying that $A \not\in A'$, which is a contradiction.

Now we shall assume that $A'(i)$ is non-empty for some $i \geq t$. Then from the above lemmas, we can write the size of $A'(i)$ as

$$|A'(i)| = \binom{\ell}{i} |A^*(i)|$$

where

$$A^*(i) = \{A \cap [\ell+2,n] : A \in A'(i)\}$$

In words, $A^*(i)$ is the set of latter segments of sets from $A'(i)$. We can take the latter segments to be from the $\ell + 2$ position onwards since we know that $\ell + 1 \not\in A$ if $A \in A'$.

We now look at those sets which come from swapping $i$ and $\ell + 1$ in the sets of $A'(i)$, none of which lie in $A$. Let

$$B(i) = \{A_{j,t+1} : j \leq \ell, j \in A \in A'(i)\}$$

$B(i)$ can be written differently, to make clear some properties and to make counting its size easier:

$$B(i) = \{B : |B \cap [1,\ell]| = i-1, \ell + 1 \in B, B \cap [\ell+2,n] \in A^*(i)\}$$

So, like with $A'(i)$, we get

$$|B(i)| = \binom{\ell}{i-1} |A^*(i)|$$

Now the aim is to construct a new $t$-intersecting family of sets using the non-empty $A'(i)$, which is larger than $A$ and has the property that it is invariant under permutations till $\ell + 1$.

It is clear that the sets that are “missing” from $A$ and cause it to not have this property are $B(i)$. So a first attempt would be to take the union of $A$ with $B(i)$, which is certainly a larger family since $A \cap B(i) = \emptyset$ (from the previous lemma, $A_{i,t+1} \not\in A$ if $A \in A'$, which is the definition of $B(i)$). The restriction that must be verified is whether the new family is still $t$-intersecting.

We will check this by partitioning the family into three disjoint sub-families: $A', A \setminus A'$, and $B(i)$. Then the only cases that needs consideration is when one of the sets is taken from $B(i)$ and the other from $A \setminus A'$ or $A'$, and when both are taken from $B(i)$. The first of these cases is taken care of by the following lemma.
Lemma 6. If \( B \in \mathcal{B}(i) \) for any \( i \) and \( D \in \mathcal{A} \setminus \mathcal{A}' \), then \( |B \cap D| \geq t \).

Proof. Let \( B = A_{i,\ell+1} \) for some \( A \in \mathcal{A}' \) and for some \( i \in A \), since all \( B \in \mathcal{B}(i) \) are obtained by such a transposition.

If \( i \notin D \) or \( \ell + 1 \in D \), the above inequality is clearly true due to \( A \) being \( t \)-intersecting. So assume \( i \in D \) and \( \ell + 1 \notin D \).

\[
\begin{array}{c|c|c}
A & i & \ell + 1 \\
\hline
D_{i,\ell+1} & 0 & 1 \\
\end{array}
\]

We will prove the contrapositive. Assume \( A_{i,\ell+1} \cap D \neq \emptyset \), i.e. \( A_{i,\ell+1} \cap D \) contains one less element (which is \( i \)) than \( A \cap D \), and so we get \( |A_{i,\ell+1} \cap D| \geq t \), i.e. \( |B \cap D| \geq t \).

For the second and third cases, we need a stronger result: we must find a condition which guarantees that two sets from \( A' \) have an intersection size of at least \( t + 1 \) (like in the previous lemma’s proof with \( A \) and \( D \)), so that on transposing some element with \( \ell + 1 \) (which reduces the intersection by 1) to get \( B \), the intersection size is still at least \( t \). This is not true for all sets of \( A' \), and the condition is provided by the next lemma.

Lemma 7. Let \( A_1, A_2 \in \mathcal{A}' \). If \(|A_1 \cap [\ell]| + |A_2 \cap [\ell]| \neq \ell + t\), then \(|A_1 \cap A_2| \leq t + 1\).

Proof. We will prove the contrapositive. Assume \(|A_1 \cap A_2| \leq t\), which implies \(|A_1 \cap A_2| = t\) as \( A \) is \( t \)-intersecting. We must prove that \(|A_1 \cap [\ell]| + |A_2 \cap [\ell]| = \ell + t\).

Suppose the opposite is true, so that \(|A_1 \cap [\ell]| = \ell - x \) and \(|A_2 \cap [\ell]| = t' + x \) where \( t' \neq t \). Since we have assumed \(|A_1 \cap A_2| = t\), we know

\[ t' < t \]

First we permute \( A_1 \) and \( A_2 \) so to get \( A_1^* \) and \( A_2^* \), such that \( A_1^* \) has all the 1s in \([1, \ell]\) in a contiguous initial segment, while \( A_2^* \) has the same for 0s. Now \( A_1^*, A_2^* \in \mathcal{A}' \) since \( \mathcal{A}' \) is invariant under permutations of \([1, \ell]\).

Clearly \(|A_1 \cap A_2| = t\) (2 is \( t \)-intersecting).

Now since \( A_1^* \) and \( A_2^* \) have intersection size less than \( t \) in the first \( \ell \) elements, there must exist \( i > \ell + 1 \) such that \( i \in A_1^* \cap A_2^* \).

\[
\begin{array}{c|c|c}
A_1^* & i & \ell + 1 \\
\hline
A_2^* & 0 & 1 \\
\end{array}
\]

Clearly \(|A_1 \cap A_2| \geq |A_1^* \cap A_2^*| = t \) (2 is \( t \)-intersecting).

Now since \( A_1^* \) and \( A_2^* \) have intersection size less than \( t \) in the first \( \ell \) elements, there must exist \( i > \ell + 1 \) such that \( i \in A_1^* \cap A_2^* \).

\[
\begin{array}{c|c|c}
A_1^* & i & \ell + 1 \\
\hline
A_2^* & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
L_{\ell+1,i}(A_1^*) & i & \ell + 1 \\
\hline
A_2^* & 0 & 1 \\
\end{array}
\]
As $A$ is left compressed, $L_{\ell+1,i}(A_1^*) \in A$. As $i \notin L_{\ell+1,i}(A_1^*) \cap A_2^*$, we get that $|L_{\ell+1,i}(A_1^*) \cap A_2^*| = t - 1$, a contradiction. 

This lemma tells us that $B(i)$ is $t$-intersecting if $i \neq (\ell + t)/2$, since then the “source” sets from $A'(i)$ will be $(t + 1)$-intersecting. Permuting to get the sets in $B(i)$ can at most reduce the intersection of two sets by two. Further the intersection also increases by one, due to $\ell + 1$ being in all sets of $B(i)$, thus making the intersection at least $t$.

From this, we also know that if sets from both $B'(i)$ and $A'(\ell + t - i)$ are present in the same family, we are not guaranteed that it will be $t$-intersecting. So in our construction we will exclude $A'(\ell + t - i)$.

It is possible that $|A'(\ell + t - i)| > |B(i)|$, in which case the new family will not be larger. But then $|A'(i)|$ should be less than $|B(\ell + t - i)|$, so that the family formed by joining $B(\ell + t - i)$ to $A$ and removing $A'(i)$ should be larger than $A$. Note that here the above assumption of $i \neq (\ell + t)/2$ is implicit, in order to have two distinct families.

In the previous three paragraphs, we have summarised the assumption and the construction of a larger family. Now we will put it down fully.

**Case 1: $i \neq (\ell + t)/2$, $i \geq t$**

Let

$$
H_1 = (A \setminus A'(\ell + t - i)) \cup B(i)
$$

$$
H_2 = (A \setminus A'(i)) \cup B(\ell + t - i)
$$

Now we claim that

$$
\max\{|H_1|, |H_2|\} > |A|
$$

(3)

From (1) and (2) and using the same derivation we get,

$$
A'(i) = \binom{\ell}{i} |A^*(i)|
$$

$$
B(i) = \binom{\ell}{i - 1} |A^*(i)|
$$

$$
A'(\ell + t - i) = \binom{\ell}{\ell + t - i} |A^*(\ell + t - i)|
$$

$$
B(\ell + t - i) = \binom{\ell}{\ell + t - i - 1} |A^*(\ell + t - i)|
$$

Using the fact that $A$, $B(i)$, and $B(\ell + t - i)$ are disjoint, we can write

$$
|H_1| = |A| - |A'(\ell + t - i)| + |B(i)|
$$

$$
|H_2| = |A| - |A'(\ell + t - i)| + |B(\ell + t - i)|
$$

If (3) is false, we would have $|H_1| < |A|$ and $|H_2| < |A|$, i.e

$$
|B(i)| < |A'(\ell + t - i)|
$$
and
\[ |B(\ell + t - i)| < |A'(i)| \]

By writing out this in terms of binomial coefficients, \(|A^*(i)|\), and \(|A^*(\ell + t - i)|\) and multiplying the two inequalities, we get (after simplification):
\[ i(\ell + t - i) \leq (i + 1 - t)(\ell - i + 1) \]

which is false since \(t \geq 2\), so that
\[ i > i + 1 - t \quad \text{and} \quad \ell + t - i > \ell - i + 1 \]

Thus the case where \(i \neq (\ell + t)/2\) is complete, giving that \(A'(i) = \emptyset\) for \(i \neq (\ell + t)/2\).

**Case 2: \(i = (\ell + t)/2\)**

As before, we assume \(A'(i) \neq \emptyset\). So we have
\[ |A'(i)| = \binom{\ell}{i} |A^*(i)| \]

Now from the previous case we know that we cannot add the whole of \(B(i)\) to \(A\) or to \(A \setminus A'(i)\) without breaking \(t\)-intersection, as we can have two sets from \(B(i)\) which intersect in less than \(t\) positions.

So we consider a subset of \(B(i)\), which will remain \(t\)-intersecting. We do so by considering the biggest subset \(D\) of \(A^*(i)\), with the property that every set of \(D\) contains a common element \(d > \ell + 1\). Then if we only add the subset of \(B(i)\) whose sets have their latter halves from \(D\) instead of \(A^*(i)\), \(t\)-intersection will be maintained, we can show that this family is sufficiently big when combined with a particular subset of \(A\). We will denote this subset of \(B(i)\) by \(G\):

\[ G = \{ B : B \in B(i), B \cap [\ell + 2, n] \in D \} \]

or equivalently,
\[ G = \{ B : |B \cap [1, \ell]| = i - 1, \ell + 1 \in B, B \cap [\ell + 2, n] \in D \} \]

To get a lower bound on \(|D|\) we use the pigeonhole principle. By definition, every set in \(A^*(i)\) contains \(k - i = k - \frac{\ell + t}{2}\) elements. So,
\[ |D| \geq \frac{k - i}{n - \ell - 1} |A^*(i)| \]  \hspace{1cm} (4)

Because the sets in \(D\) are guaranteed a common element in \([\ell + 2, n]\), we can retain those sets of \(A'(i)\) which also contain this common element without losing \(t\)-intersection in the new family. So define our new family \(H\) as
\[ H = (A \setminus A'(i)) \cup G \]

where \(A'_1(i)\) is defined as:
\[ A'_1(i) = \{ A \in A'(i) : (A \cup [\ell + 2, n]) \notin D \} \]

Like before, \(G\) and \(A\) are disjoint, and \(H\) is \(t\)-intersecting. So for \(|H| > |A|\), we need
\[ |G| > |A'_1(i)| \]  \hspace{1cm} (5)
Now we write out the sizes of the constituent families:

\[ |G| = \binom{\ell}{i-1} |A^*(i)| \]

\[ |A'_i(i)| = \binom{\ell}{i} (|A^*(i)| - |D|) \]

Substituting in (5) and simplifying, the sufficient condition is

\[ \binom{\ell+1}{i} |D| > \binom{\ell}{i} |A^*(i)| \]

But substituting the lower bound on \( D \) from (4), a stronger sufficient condition is that

\[ \binom{\ell+1}{i} \frac{k-i}{n-\ell-1} > \binom{\ell}{i} \]

On simplifying this and using that \( \ell = (\ell + t)/2 \), we get

\[ n < (k-t+1) \left( 2 + \frac{2(t-1)}{\ell-t+2} \right) \]

This would be satisfied if

\[ n < (k-t+1) \left( 2 + \frac{(t-1)}{r} \right) \]  

(6)

where \( r \geq (\ell - t + 2)/2 \), i.e. \( \ell \leq t + 2r - 2 \). The statement of the lemma guarantees (6) and this bound on \( \ell \) is guaranteed since \( \ell < t + 2r \) was assumed, and \( \ell - t \) is even as \( \ell + t \) is even.

Hence we have shown that the sufficient condition for \(|H| > |A|\) is true, and so \( A' \left( \frac{\ell+t}{2} \right) = \emptyset \).

Note that in both of the previous cases, the new family constructed has been invariant under permutations till the \((\ell + 1)\)th position.

Case 3: \( i < t \)

This case can be dealt with by making use of the lemmas that arose naturally in case 1. We will show that if \( A'(i) \neq \emptyset \), and if \( j \in A_1 \in A'(i), A^*_1 = (A_1)_{j,\ell+1} \) will have an intersection size of \( t \) with all the sets of \( A \). But since from Lemma 5 \( A^*_1 \notin A \), this is a contradiction since \( |A \cup A^*_1| > |A| \), which was assumed maximum.

Let \( A_2 \in A \setminus A' \). Then from Lemma 6, \( |A^*_1 \cap A_2| \geq t \). On the other hand if \( A_2 \in A' \), we know that \( |A_1 \cap [1, \ell]| = i < t \) (by definition of \( A'(i) \)) and \( |A_2 \cap [1, \ell]| \leq \ell \), so the conditions of Lemma 7 are satisfied. Therefore \( |A^*_1 \cap A_2| = |A_1 \cap A_2| - 1 \geq t \).

Hence \( A \cup A^*_1 \) is \( t \)-intersecting, and so \( A'(i) = \emptyset \) for \( i < t \).

Therefore \( A'(i) = \emptyset \) for all \( i \leq \ell \).

Thus the main lemma is proved.

Now we will use this lemma in proving the actual theorem, using the approach outlined earlier.
Proof of Main Theorem.

Case 1: \( n = 2k, t = 1 \)
This case follows from the Erdős-Ko-Rado Theorem, or by this simple argument: Clearly \( M(2k,k,1) \geq \binom{2k-1}{k-1} \), and at the same time, since an intersecting family cannot contain a set and its complement,

\[
M(2k,k,1) \leq \frac{1}{2} \binom{2k}{k} = \frac{1}{2} \left( \binom{2k-1}{k} + \binom{2k-1}{k-1} \right) = \binom{2k-1}{k-1}
\]

Case 2:
Here we have that

\[
(k - t + 1) \left( 2 + \frac{t - 1}{r + 1} \right) < n < (k - t + 1) \left( 2 + \frac{t - 1}{r} \right)
\]

where \( r \) is the largest integer for which this is true, and we also have that \( A \) is left-compressed. Hence from the main lemma (Lemma 2) we get that \( A \) is invariant under permutations in \([1, t + 2r]\).

Now we observe that \( \bar{A} \) is right-compressed, contains subsets of \([n]\) is size \( n - k \), and is \( n - 2k + t \) intersecting. We also easily see that \(|A| = |\bar{A}| = M(n, k, t) = M(n, n - k, n - 2k + t)\).

By expanding and multiplying, we can get, using the condition on \( n \) in this case, that

\[
n < (k' - l' + 1) \left( 2 + \frac{t' - 1}{r'} \right)
\]

where \( k' = n - k, t' = n - 2k + t, \) and \( r' = k - t - r, \) and \( r' \) is the maximum integer for which this is true. Hence the condition for the “dual” (i.e. reformulated for right compressed sets) of the main lemma holds for this dual maximal family, which gives that \( \bar{A} \) is invariant under permutations in \([n - t' - 2r' + 1, n] = [t + 2r + 1, n]\).

This same result must hold for \( A \), i.e. it must be invariant under permutation in both \([1, t + 2r]\) and \([t + 2r + 1, n]\). Now using that \( A \) is left compressed and that \( n > 2k - t \), we can prove the next lemma:

Lemma 8. If \( A_1, A_2 \in A \) with \( A \) left compressed and invariant under permutations in \([1, t + 2r]\) and in \([t + 2r + 1, n]\), then

\[
|A_1 \cap A_2 \cap [1, t + 2r]| \geq t
\]

Proof. Suppose the lemma is false. Then from the family of all pairs of sets which do not obey the inequality of the lemma, pick \( A_1, A_2 \) to be such that their intersection size is minimum.

Now permute both sets such that all intersection points in \([1, t + 2r]\) form a contiguous initial segment, and all intersection points in \([t + 2r + 1, n]\) form a contiguous final segment. Call these sets \( A_1^s \) and \( A_2^s \). Clearly neither the intersection size nor the intersection size in the first \( t + 2r \) positions is changed.

\[
\begin{array}{ccccccc}
A_1 & 1 & \cdots & t+2r & n \\
A_2 & 1 & \cdots & t+2r & n \\
A_1^s & 1 & 111 & \cdots & 1 \\
A_2^s & 1 & 111 & \cdots & 1 \\
\end{array}
\]
Since $\mathcal{A}$ is $t$-intersecting, and the number of intersection points is less than $t$ in $[1, t + 2r]$, we must have that $n \in A_1^* \cap A_2^*$ by construction.

If there is an $i$ s.t. $i \notin A_1^* \cup A_2^*$, then the pair of sets $L_{i,n}(A_1^*)$, $A_2$ will contradict the minimality of intersection of $A_1$ and $A_2$.

If there is no $i$ s.t. $i \notin A_1^* \cup A_2^*$, then $n = |A_1^* \cup A_2^*| = |A_1^*| + |A_2^*| - |A_1^* \cap A_2^*| \geq 2k - t$, which contradicts that $n > 2k - t$.

From this lemma we get that $\mathcal{A} \subseteq \mathcal{F}(r)$. But since $\mathcal{A}$ is invariant under permutations in $[1, t + 2r]$, $\mathcal{A}$ is exactly $\mathcal{F}(r)$.

Case 3: $n = (k - t + 1) \left(2 + \frac{t - 1}{r + 1}\right)$

Proceeding as in case 2, we can get the following inequality:

$$|A_1 \cap A_2 \cap [1, t + 2r + 2]| \geq t$$

where $A_1, A_2 \in \mathcal{A}$. This combined with the main lemma tells us that $\mathcal{A}$ is $\mathcal{F}(r)$ or $\mathcal{F}(r + 1)$.

Hence the theorem is proved.

References


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