

Descent basis of Δ -Springer modules $R_{n,\lambda,s}$

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JOINT WORK WITH Raymond Chou (UCSD)

PUNCHLINE: We give a construction for a monomial basis of $R_{n,\lambda,s}$
that generalizes known descent bases!!

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NOTATION All tableaux will be in French notation...

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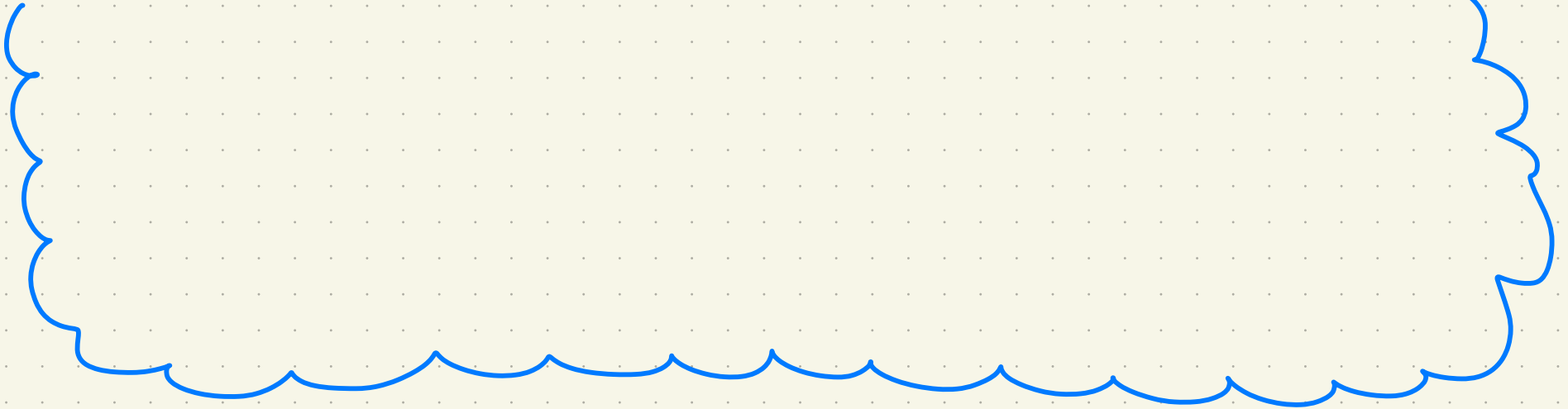
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$\Rightarrow$ The constructions give direct connections between structure of rings
+ Schur expansion of graded Frobenius

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$$\text{Frob}(V) = \sum_{\lambda} c_{\lambda} s_{\lambda}(\underline{x})$$

SCHUR FUNCTIONS

$\Downarrow$

$\text{Frob}(V)$ is SCHUR POSITIVE
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GRADED
VERSION

$$V = \bigoplus_{d \geq 0} V_d \quad \dots \text{graded } S_n\text{-mod}$$

$$\text{Frob}_q(V) = \sum_{d \geq 0} q^d \text{Frob}(V_d)$$

• COINVARIANT RING

$\mathbb{Q}[\underline{x}] = \mathbb{Q}[x_1, \dots, x_n]$ has natural S_n action (permuting the variables)

Def The coinvariant ring (of Type A_{n-1}) is

$$R_n := \mathbb{Q}[\underline{x}] / \langle e_1(\underline{x}), \dots, e_n(\underline{x}) \rangle \quad \text{with } e_k(\underline{x}) = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}$$

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$\cdot \text{Frob}_q(R_n) = \sum_{S \in \text{SST}_n} q^{\text{cocharge}(S)} S_{\text{shape}(S)}$

$\cdot \text{Hilb}_q(R_n) = [n]_q!$

• WHAT IS COCHARGE??

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$$cc(rw(S)) =$$

Steps

• Label 1 with 0

• If we label i with k , we label $i+1$ with

$$\begin{cases} k & \text{if } i \dots i+1 \text{ (} i+1 \text{ is to the RIGHT of } i \text{)} \\ k+1 & \text{if } i+1 \dots i \text{ (} i+1 \text{ is to the LEFT of } i \text{)} \end{cases}$$

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$$= 8$$

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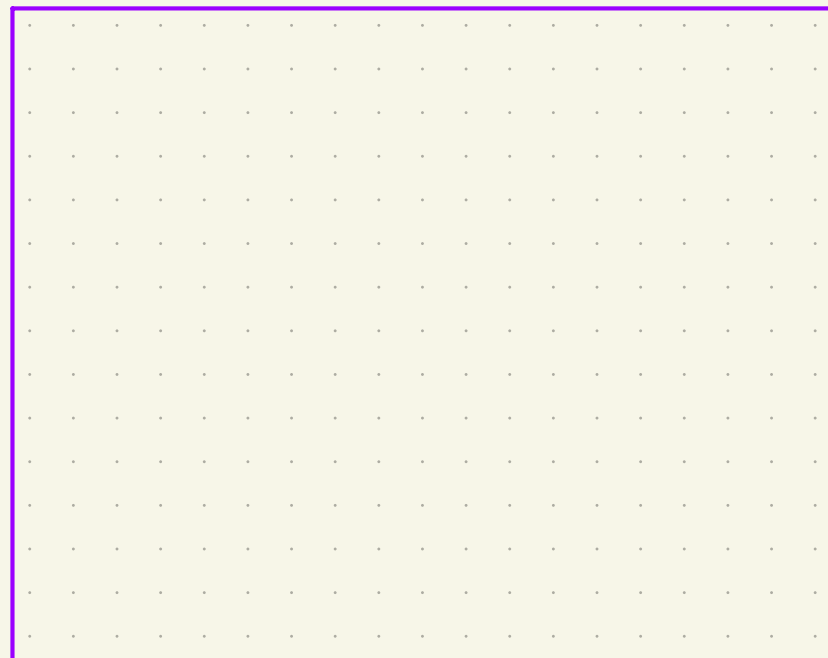
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FACTS

• if $P(w) = P(w')$,

then $cocharge(w) = cocharge(w')$

• We can also construct $cc(S)$ directly

$$cc(S) =$$

3		
1	2	
0	1	1

$$\text{Frob}_q(R_n) = \sum_{S \in \text{SYT}_n} q^{\text{cocharge}(S)} S_{\text{shape}(S)}$$

ex $n=3$ $\mathbb{Q}[x_1, x_2, x_3] / \langle x_1 + x_2 + x_3, x_1 x_2 + x_1 x_3 + x_2 x_3, x_1 x_2 x_3 \rangle$

Four SYT:

1	2	3
---	---	---

,

3	
1	2

,

2	
1	3

,

3
2
1

$\Downarrow \text{cc}(S)$

0	0	0
---	---	---

,

1	
0	0

,

1	
0	1

,

2
1
0

Then $\text{Frob}_q(R_3) = S_{\square} + (q + q^2) S_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}} + q^3 S_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}}$

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• ARTIN basis

$$\left\{ f_\sigma(\underline{x}) := \prod_{\substack{i < j \\ \sigma_i > \sigma_j}} x_{\sigma_i} : \sigma \in S_n \right\}$$

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↓

- descent basis of R_n
- descent bases of 2 different generalizations---

Two families of rings--

GARSIA-PROCESI (1992)

- indexed by partitions $\lambda \vdash n$

HAGLUND-RHOADES-SHIMOZONO (2016)

- indexed by (n, k) , where $k \leq n$

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$$R_\lambda := \mathbb{Q}[x] / \langle \text{ed}(s) \mid s \subseteq \{x_1, \dots, x_n\} \mid |s| - \underline{p}_{|s|}^n(\lambda) < d \rangle$$

of boxes that are NOT in first

$(n - |s|)$ columns of λ

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- generalization of R_n
- $R_{n,n} = R_n$

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$$\simeq_{S_n} \mathbb{Q} \left[\frac{s_n}{s_{\lambda_1} x \dots x s_{\lambda_\ell}} \right]$$

$$\simeq_{S_n} H^*(\mathcal{F}_\lambda)$$

Springer fiber corr. to
nil. matrices of Jordan type λ

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- generalization of R_n

$$R_{n,n} = R_n$$

$$\simeq_{S_n} \mathbb{Q} [OP_{n,k}]$$

Ordered set partition of n into
 k nonempty blocks

$$\simeq_{S_n} H^*(\underline{X}_{n,k}) \quad (\text{Rhoades-Pauluski})$$

space of spanning
line arrangements

GARSIA-PROCESI

• $q=0$ of modified macdonald

• $\text{Frob}_q(R_\lambda) = \tilde{H}_\lambda[x; q]$
(modified Hall-Littlewood)

$$= \sum_{T \in \text{SSYT}(\lambda)} q^{\text{cocharge}(T)} S_{\text{shape}(T)}$$

HAGLUND-RHOADES-SHIMOZONO

• $t=0$ of Delta Conjecture * UP TO TWIST

• $\text{Frob}_q(R_{n,k})$

$$= \sum_{S \in \text{ST}_n} q^{\text{maj}(S)} \begin{bmatrix} n - \text{des}(S) - 1 \\ n - k \end{bmatrix}_q S_{\text{shape}(S)}$$

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Q2 USING THIS, CAN WE GET A
DESCENT BASIS?

GARSIA-PROCESI

$$\text{Thm (LASCAUX)} \quad \text{Frob}_q(R_\lambda) = \sum_{S \in \Sigma_\lambda} q^{\text{cocharge}(S)} S_{\text{shape}(S)}$$

$$\text{where } \Sigma_\lambda = \{ S \in \text{SYT}_n, \text{ctype}(S) \triangleq \lambda \} \subseteq \text{SYT}_n$$

To any $S \in \text{SYT}_n$, we can associate a partition of n called its "catabolizability type"

GARSIA-PROCESI

Thm
(Lascoux)
$$\text{Frob}_q(R_\lambda) = \sum_{S \in \Sigma_\lambda} q^{\text{cocharge}(S)} S_{\text{shape}(S)}$$

where $\Sigma_\lambda = \{ S \in \text{SYT}_n, \text{ctype}(S) \triangleq \lambda \} \subseteq \text{SYT}_n$

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FACTS

- (Lascoux) There exists a cocharge/shape preserving bijection

$$\text{std}: \text{SSYT}(\lambda) \xrightarrow{\sim} \Sigma_\lambda$$

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GARSIA-PROCESI

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- concrete algorithm to compute $\text{ctype}(w) := \text{ctype}(P(w))$ (Blasiak)

GARSIA-PROCESI

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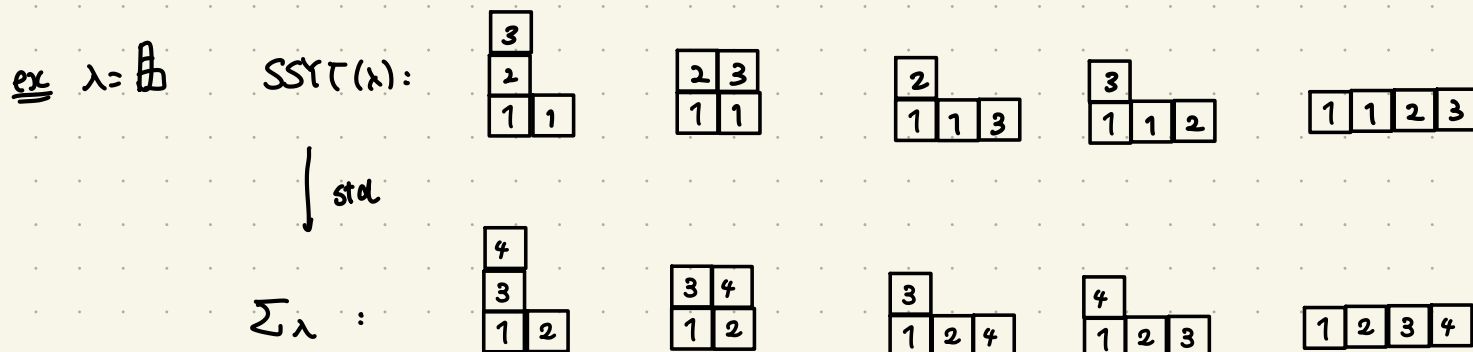
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DESCENT BASIS

Thm (H. 2024)

$\{ \underline{x}^{\text{rev}(\text{cc}(w))} : w \text{ st } P(w) \in \Sigma_\lambda \}$ is a monomial basis of R_λ

ex $R_{\mathbb{A}}$ We know $\Sigma_{\mathbb{A}} =$

4
3
1 2

,

3 4
1 2

,

3
1 2 4

,

4
1 2 3

,

1 2 3 4

 }

There are 3 different $w \in S_n$ st $P(w) =$

4
3
1 2

 $= S'$

DESCENT BASIS

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4
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 $= S'$

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4312

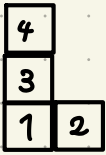
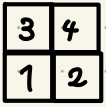
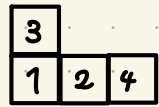
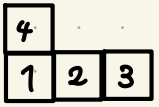
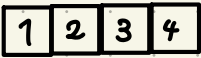
4132

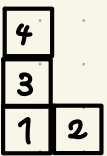
1432

DESCENT BASIS

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$P(w) = S$

4312

4132

1432

$cc(w)$

2100

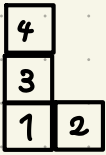
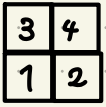
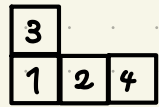
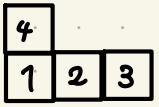
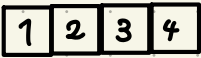
2010

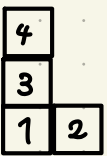
0210

DESCENT BASIS

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$P(w) = S$

4312

4132

1432

$cc(w)$

2100

2010

0210

$\text{rev}(cc(w))$

0012

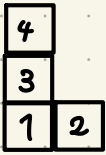
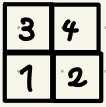
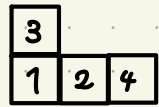
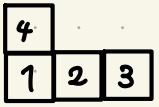
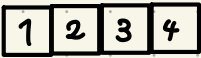
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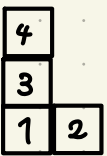
0120

DESCENT BASIS

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$P(w) = S$

4312

4132

1432

$cc(w)$

2100

2010

0210

$\text{rev}(cc(w))$

0012

0102

0120

$\underline{x}^{\text{rev}(cc(w))}$

$x_3 x_4^2$

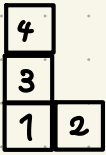
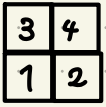
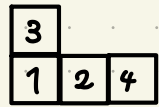
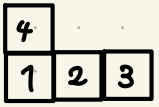
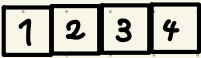
$x_2 x_4^2$

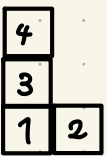
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DESCENT BASIS

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$P(w) = S$

$cc(w)$

$\text{rev}(cc(w))$

$\underline{x}^{\text{rev}(cc(w))}$

4312

2100

0012

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4132

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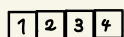
1432

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0120

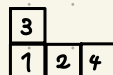
$x_2 x_3^2$

BASIS = $\{ 1, x_2, x_3, x_4, x_2 x_4, x_3 x_4, x_1 x_4, x_1 x_3, x_2 x_3, x_2 x_3^2, x_2 x_4^2, x_3 x_4^2 \}$







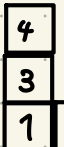
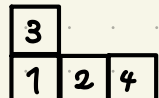
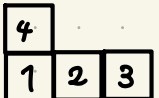
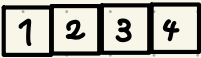


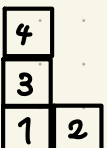


DESCENT BASIS

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$P(w) = S$

$cc(w)$

$\text{rev}(cc(w))$

$\underline{x}^{\text{rev}(cc(w))}$

4312

2100

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$x_3 x_4^2$

4132

2010

0102

$x_2 x_4^2$

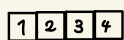
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0210

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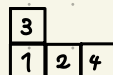
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$\text{Frob}_q(R_{\mathbb{A}}) = S_{\mathbb{A}} + q S_{\mathbb{A}} + q^2 S_{\mathbb{A}} + q^2 S_{\mathbb{A}} + q^3 S_{\mathbb{A}}$

DESCENT BASIS

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$\{ \underline{x}^{\text{rev}(\text{cc}(w))} : w \text{ st } P(w) \in \Sigma_\lambda \}$ is a monomial basis of R_λ

UPSHOTS · Compatible with $\text{Hilb}(R_\lambda)$

$$\text{Hilb}_q(R_\lambda) = \sum_{s \in \Sigma_\lambda} q^{\text{cocharge}(s)} \dim(V_{\text{shape}(s)}) = \sum_{\substack{w \in S_n \\ P(w) \in \Sigma_\lambda}} q^{\text{cocharge}(w)}$$

DESCENT BASIS

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$\{ \underline{x}^{\text{rev}(\text{cc}(w))} : w \text{ st } P(w) \in \Sigma_\lambda \}$ is a monomial basis of R_λ

UPSHOTS · Compatible with $\text{Hilb}(R_\lambda)$

$$\text{Hilb}_q(R_\lambda) = \sum_{S \in \Sigma_\lambda} q^{\text{cocharge}(S)} \dim(V_{\text{shape}(S)}) = \sum_{\substack{w \in S_n \\ P(w) \in \Sigma_\lambda}} q^{\text{cocharge}(w)}$$

· Can use to show $\text{Frob}_q(R_\lambda) = \tilde{H}_\lambda[x; q]$ * using $\text{Frob}(R_\lambda) = h_\lambda$

(via: determining expression for $\langle e_\lambda, \text{Frob}_q(R_\lambda) \rangle$ $\forall \lambda \vdash n$.
by constructing basis of S_0 -antisymm. component of R_λ)

DESCENT BASIS

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REMARK

Carlsson - Chu have a different construction of this in terms of shuffles:

ex $R_{\mathbb{A}}$ consider $\mathcal{D}_3 = \{ 000, 001, 010, 101, 011, 012 \}$ $\mathcal{D}_1 = \{ 0 \}$

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HARD TO COUNT THIS SET...

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1	2	3	4
---	---	---	---

4
1 2 3

3	4
1	2

4
3
1 2

3
1 2 4

DESCENT BASIS

Thm (H. 2024)

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1	2	3	4
---	---	---	---

4		
1	2	3

3	4
1	2

4	
3	
1	2

3		
1	2	4

Thm (H. 2024) $D_\lambda = \{ \text{rev}(\text{cc}(w)) : w \text{ st } P(w) \in \Sigma_\lambda \}$

HAGLUND - RHOADES - SHIMOZONO

RECALL

$$F_{\text{rob } q}(R_{n,k}) = \sum_{\substack{S \in \text{ST}_n \\ \text{des}(S) < k}} q^{\text{maj}(S)} \begin{bmatrix} n - \text{des}(S) - 1 \\ n - k \end{bmatrix}_q S_{\text{shape}(S)}$$

HAGLUND - RHOADES - SHIMOZONO

RECALL

$$F_{\text{rob } q}(R_{n,k}) = \sum_{\substack{S \in \text{ST}_n \\ \text{des}(S) < k}} q^{\text{maj}(S)} \underbrace{\begin{bmatrix} n - \text{des}(S) - 1 \\ n - k \end{bmatrix}_q}_{\text{gen. function}} S_{\text{shape}(S)}$$

$\text{maj} \rightarrow \text{cocharge}$

"Folklore conjecture"

(K. Chan's Thesis)

2024

$$\sum q^{|\mu|}$$

$$\mu \leq (n-k) \times (k - \text{des}(S) - 1)$$

HAGLUND - RHOADES - SHIMOZONO

RECALL

$$\text{Frob}_q(R_{n,k}) = \sum_{\substack{S \in \text{ST}_n \\ \text{des}(S) < k}} q^{\text{maj}(S)} \underbrace{\begin{bmatrix} n - \text{des}(S) - 1 \\ n - k \end{bmatrix}_q}_{\text{gen. function}} S_{\text{shape}(S)}$$

$\text{maj} \rightarrow \text{cocharge}$
 "Folklore conjecture"
 (K. Chan's Thesis)
 2024

$$\sum q^{|\mu|}$$

$$\mu \leq (n-k) \times (k - \text{des}(S) - 1)$$

$$\text{Frob}_q(R_{n,k}) = \sum_{(S, \mu) \in \Sigma_{n,k}} q^{\text{cocharge}(S) + |\mu|} S_{\text{shape}(S)}$$

$$\text{where } \Sigma_{n,k} = \left\{ (S, \mu) : \begin{array}{l} S \in \text{ST} \quad \text{des}(S) < k \\ \mu \leq (n-k) \times (k - \text{des}(S) - 1) \end{array} \right\}$$

HAGLUND - RHOADES - SHIMOZONO

$$\underline{\text{Thm}} \quad \text{Frob}_q(R_{n,k}) = \sum_{(S, \mu) \in \Sigma_{n,k}} q^{\text{cocharge}(S) + |\mu|} S_{\text{shape}(S)}$$

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ex $n=4, k=2$

First, find all $S \in \text{STT}_4$ with at most 1 descent

1	2	3	4
---	---	---	---

$\text{des}(S) = 0$

2			
1	3	4	

$\text{des}(S) = 1$

3			
1	2	4	

$\text{des}(S) = 1$

4			
1	2	3	

$\text{des}(S) = 1$

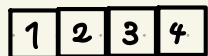
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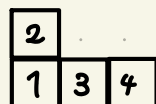
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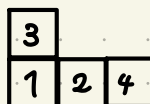
First, find all $S \in \text{STT}_4$ with at most 1 descent



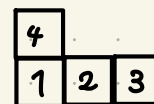
$\text{des}(S) = 0$



$\text{des}(S) = 1$



$\text{des}(S) = 1$



$\text{des}(S) = 1$

Then find $\mu \leq \underline{2 \times (2 - \text{des}(S) - 1)}$

$$\Sigma_{4,2} = \left\{ \begin{array}{l} \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}, \emptyset \right), \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}, \square \right), \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}, \square \right) \\ \left(\begin{array}{|c|} \hline 2 \\ \hline 1 & 3 & 4 \\ \hline \end{array}, \emptyset \right), \left(\begin{array}{|c|} \hline 3 \\ \hline 1 & 2 & 4 \\ \hline \end{array}, \emptyset \right), \left(\begin{array}{|c|} \hline 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \emptyset \right) \end{array} \right\}$$

· DESCENT BASIS

We reformulate the construction of Haglund - Rhoades - Shimozono to match our framework

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Def for any pair (w, μ) where $(P(w), \mu) \in \Sigma_{n,k}$

define $cc(w, \mu)_i := \begin{cases} cc(w)_i & \text{if } w_i \in \{1, \dots, k\} \\ cc(w)_i + \mu_{n-w_i+1} & \text{if } w_i \in \{k+1, \dots, n\} \end{cases} \rightarrow \text{boosting the cocharge of the last } (n-k) \text{ entries of } cc(w).$

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ex $n=7$
 $k=4$ $\left(\begin{smallmatrix} 45 \\ 12367 \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right) \in \Sigma_{4,2}$ (since $(n-k) \times (k - \text{des}(S) - 1) = 3 \times 2$)

Then to get $cc(4512367, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$

$w: 4512367$

$cc(w): 1100011$

$\mu: \quad \quad 1 \quad \quad 1 \quad 2$

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$\{ \underline{x}^{\text{rev}(cc(w, \mu))} : (w, \mu) \text{ st } (P(w), \mu) \in \Sigma_{n,k} \}$ is a monomial basis of $R_{n,k}$.

GARSIA-PROCESI

$$\text{Frob}_q(R_\lambda) = \sum_{S \in \Sigma_\lambda} q^{\text{cocharge}(S)} S_{\text{shape}(S)}$$

$$\Sigma_\lambda = \{ S \in \text{SYT}_n, \text{ctype}(S) \triangleq \lambda \} \subseteq \text{SYT}_n$$

$$\text{DESCENT BASIS} : \{ \underline{X}^{\text{rev}(\text{cc}(w))} : P(w) \in \Sigma_\lambda \}$$

HAGLUND-RHOADES-SHIMOZONO

$$\text{Frob}_q(R_{n,k}) = \sum_{(S,\mu) \in \Sigma_{n,k}} q^{\text{cocharge}(S) + |\mu|} S_{\text{shape}(S)}$$

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NEW RESULTS

$R_{n,\lambda,s}$

Δ -SPRINGER MODULES (GRIFFIN, 2021)

indexed by (n, λ, s) where $\begin{cases} \lambda \vdash k \leq n \\ s \geq e(\lambda) \end{cases}$

$$R_{n,\lambda,s} := \mathbb{Q}[x] / \langle x_1^s, \dots, x_n^s, e_d(s) \text{ for } |S| - \underline{p_{|S|}^n(\lambda)} < d \rangle$$

of boxes that are NOT in first $(n - |S|)$ columns of λ

$\hookrightarrow$ ordered set partition into s blocks with $|B_i| \geq 2\lambda_i$

$$\bullet \simeq_{S_n} \mathbb{Q}[\text{OSP}_{n,\lambda,s}]$$

$$\simeq_{S_n} H^*(Y_{n,\lambda,s}) \quad (\Delta\text{-Springer variety}) \quad (\text{Griffin, Levinson, Woo})$$

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SPECIALIZES TO ---

$$(n, \lambda, s) = (k, \lambda, e(\lambda))$$

$$(n, \lambda, s) = (n, 1^k, k)$$

GARSIA-PROCESI

$$R_\lambda := \mathbb{Q}[x] / \langle e_d(S) \mid S \subseteq \{x_1, \dots, x_n\}, |S| - \underline{p}_{|S|}^n(\lambda) < d \rangle$$

of boxes that are NOT in first $(n - |S|)$ columns of λ

HAGLUND-RHOADES-SHIMOZONO

$$R_{n,k} := \mathbb{Q}[x] / \langle x_1^k, \dots, x_n^k, e_{n-k+1}(\underline{k}), \dots, e_n(\underline{k}) \rangle$$

• BATTERY-POWERED TABLEAUX

Def Let $\mathcal{L}_{n,\lambda,s} := (n-k) \times s + \lambda$

A battery powered tableau of parameters (n,λ,s)

is a pair (D,B) of semistandard tableaux where

- B is of shape $(n-k) \times (s-1)$
- Total content is $\mathcal{L}_{n,\lambda,s}$.

Let $\mathcal{T}_{n,\lambda,s}^+ = \{ \text{all battery-powered tableaux of parameters } (n,\lambda,s) \}$

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ex $(n,\lambda,s) = (4, \overline{1}, 3)$

• TOTAL WEIGHT is $\overline{1} + \overline{1} = (3, 2, 1)$

• BATTERY SHAPE is $\overline{1}$

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ex $(n,\lambda,s) = (4, \mathbb{A}, 3)$

• TOTAL WEIGHT is $\mathbb{A} + \mathbb{A} = (3, 2, 1)$

• BATTERY SHAPE is $\mathbb{A}$

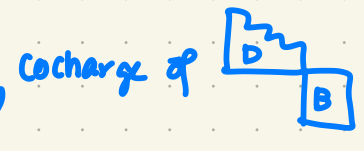
$\mathcal{T}_{(4,\mathbb{A},3)}^+ =$

The tableaux are:

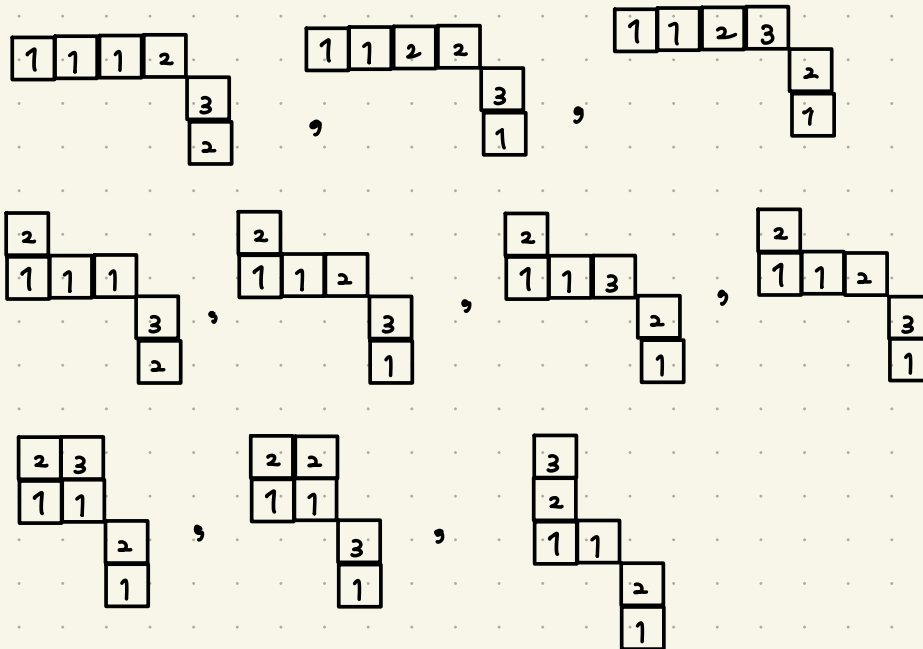
- $\begin{matrix} 1 & 1 & 1 & 2 \\ & & & 3 \\ & & & 2 \end{matrix}, \begin{matrix} 1 & 1 & 2 & 2 \\ & & & 3 \\ & & & 1 \end{matrix}, \begin{matrix} 1 & 1 & 2 & 3 \\ & & & 2 \\ & & & 1 \end{matrix}$
- $\begin{matrix} 2 \\ 1 & 1 & 1 \end{matrix}, \begin{matrix} 2 \\ 1 & 1 & 2 \end{matrix}, \begin{matrix} 2 \\ 1 & 1 & 3 \end{matrix}, \begin{matrix} 2 \\ 1 & 1 & 2 \end{matrix}, \begin{matrix} 3 \\ 1 \end{matrix}$
- $\begin{matrix} 2 & 3 \\ 1 & 1 \end{matrix}, \begin{matrix} 2 & 2 \\ 1 & 1 \end{matrix}, \begin{matrix} 3 \\ 2 \\ 1 \end{matrix}, \begin{matrix} 2 \\ 1 \end{matrix}$

Thm (Gillespie - Griffin, 2024)

$$\textcircled{\star} \text{Frob}_q(R_{n,\lambda,s}) = \frac{1}{q^{(\sum s_i) \cdot (n-k)}} \sum_{(D,B) \in T_{n,\lambda,s}^+} q^{\text{cocharge}(D,B)} S_{\text{Shape}(D)}.$$

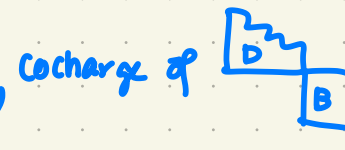


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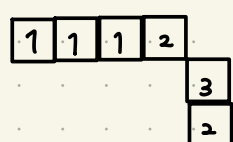
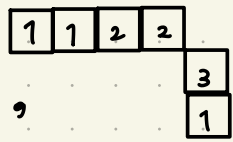
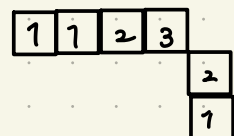


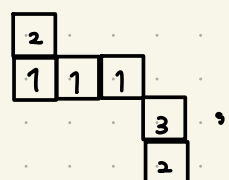
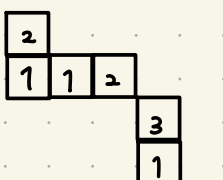
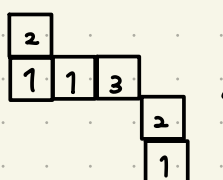
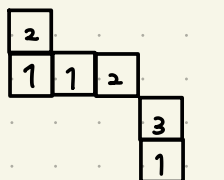
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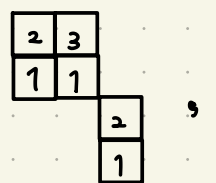
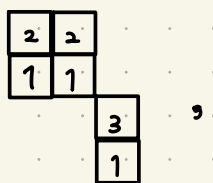
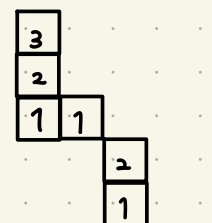
$$\textcircled{\star} \text{Frob}_q(R_{n,\lambda,s}) = \frac{1}{q^{(\frac{s+1}{2}) \cdot (n-k)}} \sum_{(D,B) \in T_{n,\lambda,s}^+} q^{\text{cocharge}(D,B)} S_{\text{Shape}(D)}.$$



ex $(n, \lambda, s) = (4, \mathbb{A}, 3)$




 $\Rightarrow (q + q^2 + q^3) S_{\mathbb{A}}$





 $\Rightarrow (q^2 + 2q^3 + q^4) S_{\mathbb{A}}$




 $\Rightarrow (q^4 + q^3) S_{\mathbb{A}} + q^4 S_{\mathbb{B}}$

$$\text{Frob}_q(R_{n,\lambda,s}) = \frac{1}{q} \left((q + q^2 + q^3) S_{\mathbb{A}} + (q^2 + 2q^3 + q^4) S_{\mathbb{A}} + (q^4 + q^3) S_{\mathbb{A}} + q^4 S_{\mathbb{B}} \right)$$

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Proof of $\textcircled{\star}$ + motivation for $T_{n,\lambda,s}^+$ is geometric...

: is there a way to understand this $\begin{cases} \text{combinatorially} \\ \text{algebraically} \end{cases}$?

NEW RESULTS

$$\underline{\text{Def}} \quad \Sigma_{n, \lambda, s} = \left\{ (S, \mu) : \begin{array}{l} \text{ctype}(S|_k) \triangleq \lambda \\ \mu \subseteq (n-k) \times (s - \text{des}(S) - 1) \end{array} \right\}$$

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note We can see that

$$\cdot \Sigma_{k, \lambda, \ell(\lambda)} = \left\{ (S, \mu) : \begin{array}{l} \text{ctype}(S|_k) \triangleq \lambda \\ \mu \subseteq (n-k) \times (s - \text{des}(S) - 1) \end{array} \right\}$$

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Thm (Chou - H. 2025+)

$\left\{ \underline{X}^{\text{rev}(\text{cc}(w, \mu))} : (P(w), \mu) \in \Sigma_{n, \lambda, s} \right\}$ is a monomial basis of $R_{n, \lambda, s}$.

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$\text{ctype}(S|_3) \in \mathbb{A}$

$\mu \leq (4-3) \times (3-\text{desc}(S)-1) \}$

$\text{SST}(\mathbb{A}) \xrightarrow{\text{std}} \Sigma_{\mathbb{A}} = \{ S \in \text{SST}_3, \text{ctype} \geq \mathbb{A} \}$

$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline \end{array}$

$\longrightarrow$

$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$

$\begin{array}{|c|} \hline 2 \\ \hline 1 & 1 \\ \hline \end{array}$

$\longrightarrow$

$\begin{array}{|c|} \hline 3 \\ \hline 1 & 2 \\ \hline \end{array}$

NEW RESULTS

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ex $(n, \lambda, s) = (4, \mathbb{A}, 3)$

$$\Sigma_{n, \lambda, s} = \left\{ (S, \mu) : \begin{array}{l} S \in \{123, \overset{1}{1}2\} \\ \mu \leq 1 \times (2 - \text{des}(S)) \end{array} \right\}$$

NEW RESULTS

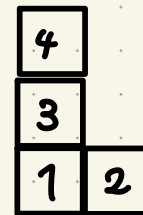
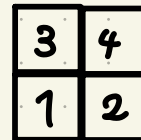
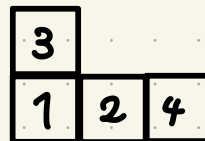
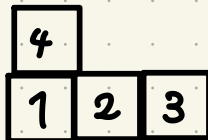
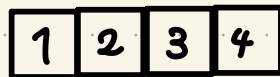
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POSSIBLE S :



POSSIBLE μ :

$\emptyset, \square, \boxplus$

$\emptyset, \square$

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$\emptyset$

NEW RESULTS

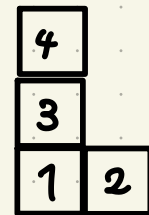
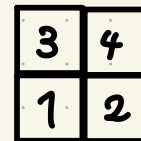
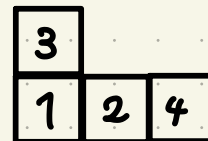
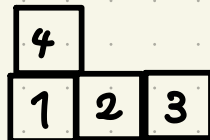
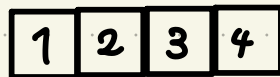
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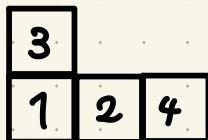
$\emptyset, \square$

$\emptyset, \square$

$\emptyset, \square$

$\emptyset$

• $P(3124) =$



NEW RESULTS

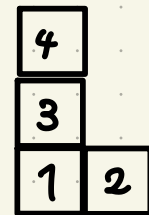
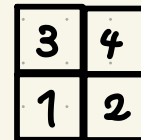
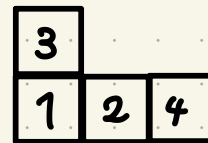
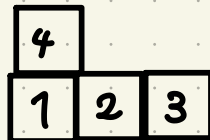
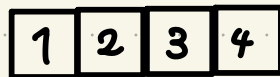
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$\emptyset, \square, \boxplus$

$\emptyset, \square$

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$\emptyset, \square$

$\emptyset$

• $P(3124) =$

3		
1	2	4

$$* cc(3124, \emptyset) = cc(3124) = 1001$$

$$* cc(3124, \square) = 1002$$

NEW RESULTS

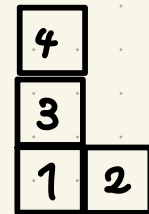
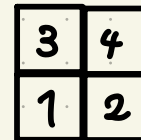
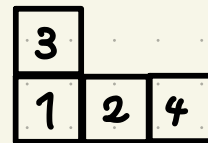
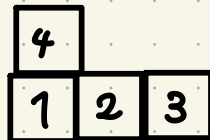
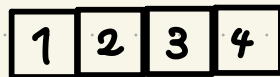
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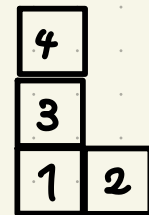
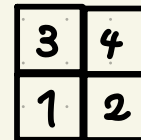
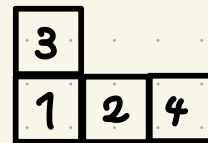
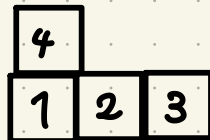
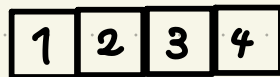
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$\Rightarrow$

$$x_1 x_4$$

$$x_1^2 x_4$$

NEW RESULTS

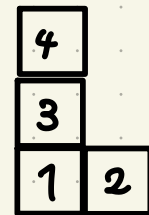
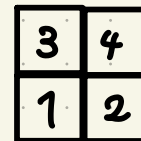
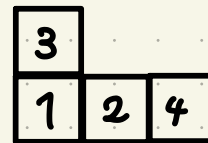
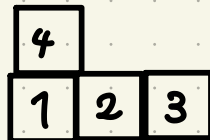
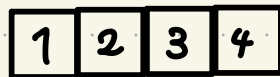
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$\emptyset, \square$

$\emptyset, \square$

$\emptyset, \square$

$\emptyset$

BASIS:

$\{$

$1, x_1, x_1^2$

x_4, x_4^2

x_3, x_3^2

x_2, x_2^2

$x_1 x_4, x_1^2 x_4$

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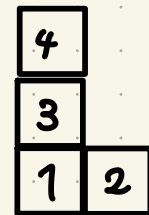
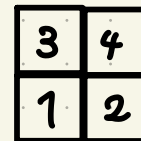
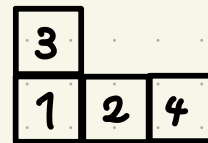
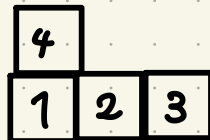
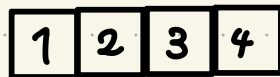
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$x_3 x_4^2$

x_3, x_3^2

$x_1 x_3, x_1^2 x_3$

$x_2 x_4, x_2^2 x_4$

$x_2 x_4^2$

x_2, x_2^2

$x_2 x_3, x_2^2 x_3$

$x_2 x_3^2$

$\}$

Note There is a shuffle ver. of this!

NEW RESULTS

Thm (Chou-H. 2025+)

$\{ \underline{X}^{\text{rev}(\text{cc}(w, \mu))} : (P(w), \mu) \in \Sigma_{n, \lambda, s} \}$ is a monomial basis of $R_{n, \lambda, s}$.

UPSHOT

- Specializes to descent bases of R_λ , $R_{n, \lambda, s}$
- Can use to derive Schur exp of $\text{Frob}_g(R_{n, \lambda, s})$

NEW RESULTS

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UPSHOT

- Specializes to descent bases of R_λ , $R_{n, \lambda, s}$
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Thm (Chou-H. 2025+)

$$\text{Frob}_q(R_{n, \lambda, s}) \simeq \sum_{(S, \mu) \in \Sigma_{n, \lambda, s}} q^{\text{recharge}(S) + |\mu|} S_{\text{shape}(S)}$$

NEW RESULTS

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Thm (Chou-H. 2025+)

$$\text{Frob}_g(R_{n, \lambda, s}) \approx \sum_{(s, \mu) \in \Sigma_{n, \lambda, s}} q^{\text{cocharge}(s) + |\mu|} S_{\text{shape}(s)}$$

$\Rightarrow$ CONNECTION TO BATTERY-POWERED TABLEAUX...

NEW RESULTS

Thm (Chou-H. 2025+)

$$\sum_{(S, \mu) \in \Sigma_{n, \lambda, s}} q^{\text{cocharge}(S) + |\mu|} S_{\text{shape}(S)} = \frac{1}{q^{\binom{s-1}{2} \cdot (n-k)}} \sum_{(D, B) \in T_{n, \lambda, s}^+} q^{\text{cocharge}(D, B)} S_{\text{shape}(D)}.$$

In particular, this gives an algebraic/combinatorial pf of the battery-powered tableaux formula!

IDEA: We have bijection $T_{n, \lambda, s}^+ \xrightarrow{\sim} \Sigma_{n, \lambda, s}$ that takes $(D, B) \mapsto (S', \mu)$ where

- $\text{shape}(D) = \text{shape}(S)$
- $\text{cocharge}(D, B) = \text{cocharge}(S) + |\mu| + \binom{s-1}{2} \cdot (n-k)$

NEW RESULTS

Thm (Chou-H. 2025+)

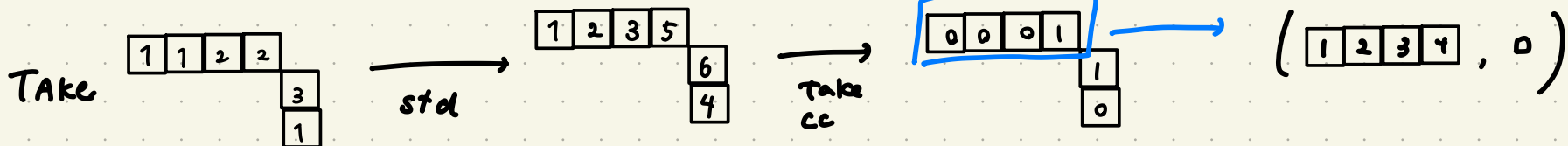
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ex



NEW RESULTS

Thm (Chou-H. 2025+)

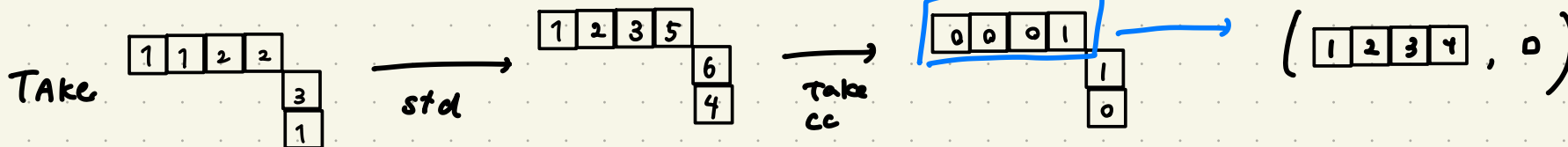
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ex



We use a similar bijection when showing our basis has the right cardinality.

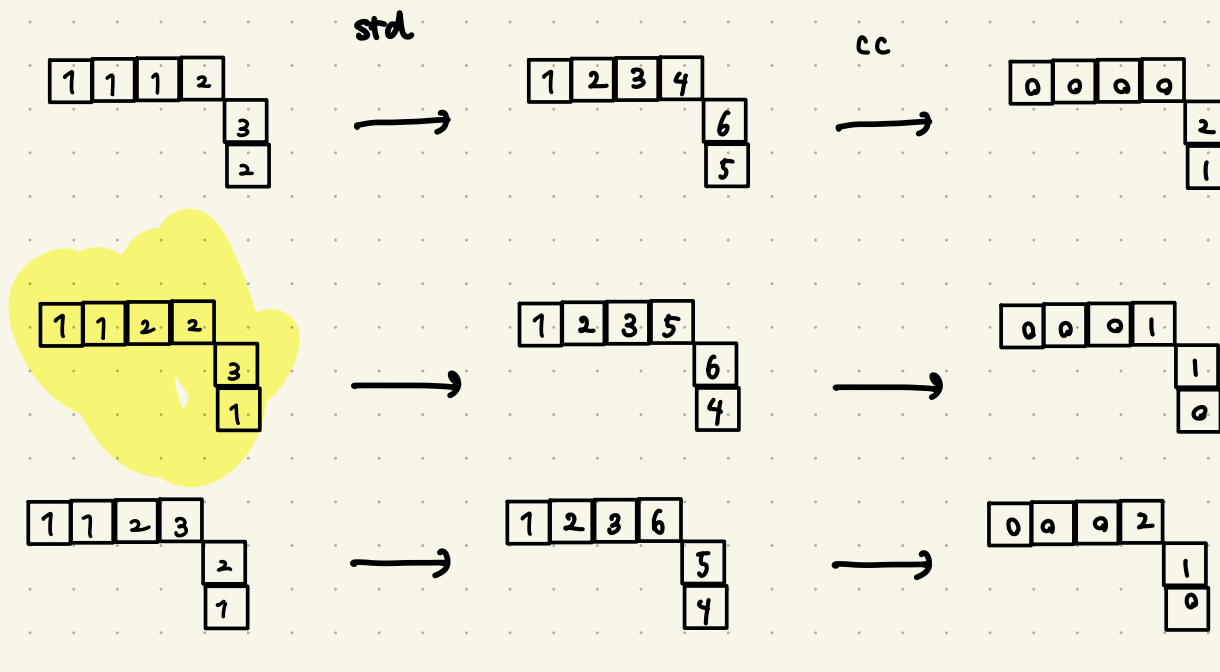
• FUTURE DIRECTIONS

- are there other bases we can construct using $\left\{ \begin{array}{l} \text{cocharge} \\ \text{tableaux} \end{array} \right\}$?
- can we use this to study the full Macdonald picture?
- higher Specht bases?

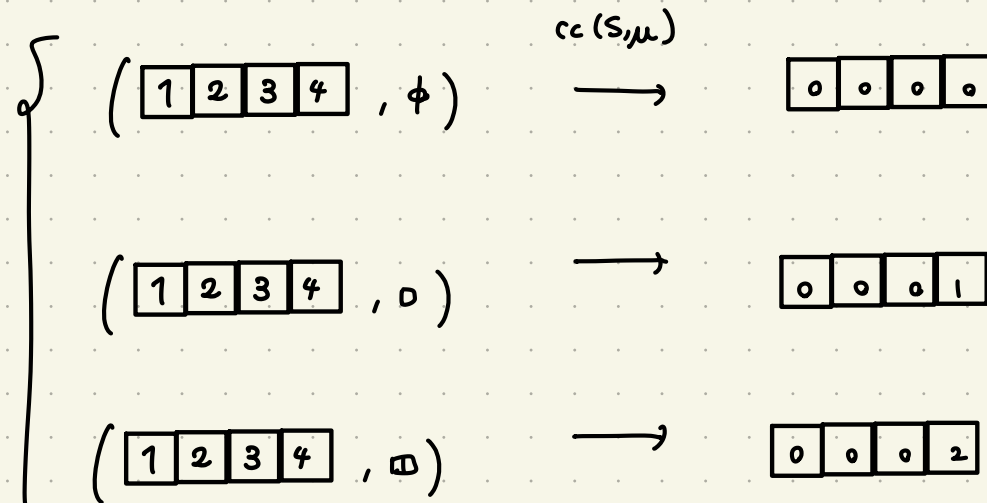
Thank You !!

MORE ex...

$\ln T_{(4, \mathbb{B}, 3)}^*$



$\ln \Sigma_{4, \mathbb{B}, 3}$



NOTE To show our basis has the right count, we use

$$\text{(or Griffin)} \quad \dim(R_{n,\lambda,s}) = \left| \bigcup_{\substack{\alpha \geq \lambda \\ (\alpha_1, \dots, \alpha_s) \models n}} \text{SSYT}(\alpha) \right| \quad (\text{follows from } R_{n,\lambda,s} \simeq \mathbb{Q}(\text{OSP}_{n,\lambda,s})).$$

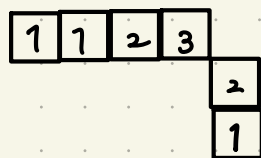
and these maps... $T_{n,\lambda,s}^+ \xrightarrow{\sim} \sum_{n,\lambda,s}$

"rip off battery"

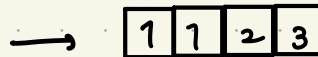
$$\bigcup_{\substack{\alpha \geq \lambda \\ (\alpha_1, \dots, \alpha_s) \models n}} \text{SSYT}(\alpha)$$

MAIN ISSUE: this doesn't behave well with cocharge! (SOURCE of MYSTERY revolving battery powered tableaux)

* The same battery may contribute different cocharge...

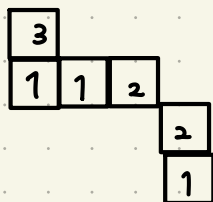


cocharge = 3

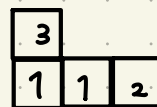


cocharge = 0

Rip off battery



cocharge = 3



cocharge = 1