A charge monomial basis of the Garsia-Procesi ring

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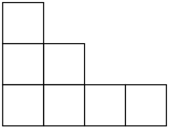
Combinatorial definitions

A *partition* $\lambda \vdash n$ is a weakly decreasing sequence of positive integers that sum to n.

The *Young diagram* of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is an left-aligned arrangement of n boxes, with λ_i many boxes in row i.

$$\lambda = (4,2,1) \vdash 7$$



Remark

The irreducible representations of S_n are indexed by partitions $\lambda \vdash n$.

Combinatorial definitions

A *standard Young tableau* (SYT) of shape $\lambda \vdash n$ is a filling of the Young diagram of λ where $\{1, \ldots, n\}$ appear exactly once and the entries are increasing within the rows and columns.

$$\begin{array}{|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 3 & 4 & 7 \\ \hline \end{array}\,.$$

The *row reading word* of T ($\mathrm{rw}(T)$) is the word we get by concatenating the row words from top to bottom.

$$\mathrm{rw}\left(\begin{array}{|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 3 & 4 & 7 \\ \hline \end{array} \right) = 6251347.$$

RSK correspondence

We denote the set of all SYT of size n as SYT_n.

Remark: RSK

By Robinson-Schensted-Knuth correspondence we have

$$S_n \Leftrightarrow \{(P,Q) \mid P,Q \in \mathrm{SYT}_n, \mathrm{shape}(P) = \mathrm{shape}(Q)\}$$

$$w \mapsto (P(w), Q(w)).$$

Coinvariant Ring

$\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathbf{x}]$ has a natural S_n action permuting the variables.

Definition

The *coinvariant ring* (of Type A_{n-1}) is

$$R_{1^n} = \mathbb{C}[\mathbf{x}]/\langle e_k(\mathbf{x}) \text{ for } \ k \in \{1, \cdots, n\}\rangle$$

where $e_k(\mathbf{x}) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$.

Remarks

R_{1^n} is isomorphic to...

- Regular rep of S_n(ungraded)
- Cohomology ring of the flag variety (as graded S_n reps)

Frobenius character

The *Frobenius characteristic map* is a map from (virtual) characters of S_n to symmetric functions of degree n (with coefficients in $\mathbb{Z}$) that takes

$$\mathrm{ch}(V_\lambda) \mapsto s_\lambda[X].$$

The resulting symmetric function $\mathrm{Frob}(V)$ is the *Frobenius character* of V. $\mathrm{Frob}(V)$ encodes the decomposition of V into irreducibles.

$$\mathrm{Frob}(V_{(2,1)} \oplus V_{(1,1,1)}) = s_{(2,1)} + s_{(1,1,1)}.$$

Graded Frobenius character

For a graded $\mathbb{C}S_n$-module $V = \oplus_{d \geq 0} V_d$, the *graded Frobenius character* $\mathrm{Frob}_q(V)$ is

$$\mathrm{Frob}_q(V) = \sum_{d \geq 0} q^d \,\mathrm{Frob}(V_d).$$

Note

For the coinvariant ring, we have

$$\mathrm{Frob}_q(R_{1^n}) = \sum_{T \in \mathrm{SYT}_n} q^{\mathrm{maj}(T)} s_{\mathrm{shape}(T)} = \sum_{T \in \mathrm{SYT}_n} q^{\mathrm{cocharge}(T)} s_{\mathrm{shape}(T)}.$$

Monomial bases of the coinvariant ring

- Artin basis

$$\{f_\sigma(\mathbf{x}) = \prod_{i<j, \sigma_i>\sigma_j} x_{\sigma_i} \mid \sigma \in S_n\}$$

Corresponds to *inversions*

$$\mathrm{inv}(\sigma) := |\{(i,j) \mid i<j, \sigma_i > \sigma_j\}|.$$

.

$\Rightarrow \deg(f_\sigma(\mathbf{x})) = \mathrm{inv}(\sigma)$.

- Descent basis

$$\{g_\sigma(\mathbf{x}) = \prod_{i, \sigma_i>\sigma_{i+1}} x_{\sigma_1} \cdots x_{\sigma_i} \mid \sigma \in S_n\}$$

Corresponds to *major index*

$$\mathrm{maj}(\sigma) := \sum_{i:\sigma_i>\sigma_{i+1}} i$$

$\Rightarrow \deg(g_\sigma(\mathbf{x})) = \mathrm{maj}(\sigma)$.

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Corresponds to *major index*

Remark

The permutation statistics inv and maj have the same distribution:

$$\sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)} = [n]_q! = \mathrm{Hilb_q}(R_{1^n}).$$

$\rightarrow$ Both bases are compatible with $\mathrm{Hilb_q}(R_{1^n})$

Garsia-Procesi Ring

Definition

For $\mu \vdash n$, the *Garsia-Procesi ring* R_μ is

$$R_\mu = \mathbb{C}[\mathbf{x}]/I_\mu$$

where the ideal I_μ is generated by

$$\{e_d(S) \mid S \subset \{x_1, \ldots, x_n\}, |S| - p^n_{|S|}(\mu) < d \leq |S|\}$$

where $p^n_k(\mu)$ is the number of boxes that are **not** in the first $(n-k)$ columns of the Young diagram of μ.

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where $p^n_k(\mu)$ is the number of boxes that are **not** in the first $(n-k)$ columns of the Young diagram of μ.

Remark

R_μ are quotients of the coinvariant ring R_{1^n} arising from geometry (Springer fibers $\subset$ Flag variety.)

$\mathrm{Frob_q}(R_\mu)$

For $\mu \vdash n$, we have

$$\mathrm{Frob_q}(R_\mu) = \tilde{H}_\mu[X;q]$$

where $\tilde{H}_\mu[X;q]$ is the *modified Hall-Littlewood polynomial.*

Theorem (Lascoux 1989)

$$\tilde{H}_\mu[X;q] = \sum_{\substack{T \in \mathrm{SYT}_n \\ \mathrm{ctype}(T^t) \trianglerighteq \mu}} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T^t)},$$

where $\trianglerighteq$ denotes the dominance order on partitions of size n and T^t denotes the transpose of T.

What is ctype(T)?

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where $\trianglerighteq$ denotes the dominance order on partitions of size n and T^t denotes the transpose of T.

Remark

The *catabolizability type* ctype(T) of a SYT is a partition.
We have a cocharge preserving bijection:

$$\{T \in \mathrm{SYT}_n \mid \mathrm{ctype}(T) \trianglerighteq \mu\} \leftrightarrow \{S \in \mathrm{SSYT} \text{ with weight } \mu\}.$$

What is charge?

For any permutation $w \in S_n$, we can construct its *charge word* $\mathrm{c}(w)$ by labeling each letter in w in the following way:

- label 1 by 0,
- if we label i by k, we label $i+1$ by
$$\begin{cases} k+1 & \text{if } i+1 \text{ is to the right of } i \\ k & \text{if } i \text{ is to the left of } i \end{cases}$$

The word consisting of these labels is the charge word $\mathrm{c}(w)$.

Example:

$$w = 4 \ \ 2 \ \ 1 \ \ 5 \ \ 3$$

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Example:

$$w = 4 \quad 2 \quad 1 \quad 5 \quad 3$$
$$\mathrm{c}(w) = 1 \quad 0 \quad 0 \quad 2 \quad 1$$

What is charge?

Definition

- charge(w) is the sum of the letters in $c(w)$.
- $\mathrm{charge}(T) = \mathrm{charge}(\mathrm{rw}(T))$.

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Remarks

- $\mathrm{maj}(\mathrm{rev}(w^{-1})) = \mathrm{charge}(w)$ (where $\mathrm{rev}(w)$ is the reverse word)
- If $P(w) = P(w')$, then $\mathrm{charge}(w) = \mathrm{charge}(w')$.

Coinvariant ring and Garsia-Procesi rings

	Coinvariant ring R_{1^n}	Garsia-Procesi R_μ
$\mathrm{Frob_q}$	$\sum_{T\in \mathrm{SYT}_n} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T^t)}$	$\sum_{\substack{T\in \mathrm{SYT}_n \\ \mathrm{ctype}(T^t)\trianglerighteq\mu}} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T^t)}$
$\mathrm{Hilb_q}$	$[n]_q! = \sum_{w\in S_n} q^{\mathrm{charge}(w)}$	$\sum_{\substack{w\in S_n \\ \mathrm{ctype}(P(w)^t)\trianglerighteq\mu}} q^{\mathrm{charge}(w)}$

Returning to monomial bases of R_{1^n}

Question

Are there subsets of monomial bases of R_{1^n} that are bases of R_μ?

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- Artin basis

$$\{f_\sigma(\mathbf{x}) = \prod_{i<j, \sigma_i > \sigma_j} x_{\sigma_i} \mid \sigma \in S_n\}$$

$$\downarrow$$

Garsia-Procesi (1992)

- Descent basis

$$\{g_\sigma(\mathbf{x}) = \prod_{i, \sigma_i > \sigma_{i+1}} x_{\sigma_1} \cdots x_{\sigma_i} \mid \sigma \in S_n\}$$

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Carlsson-Chou(2024)
"shuffles of descent monomials"

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Are there subsets of monomial bases of R_{1^n} that are bases of R_μ?

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$\downarrow$

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Remark

Neither of these bases have direct connections to $\mathrm{Frob}_q(R_\mu)$ or $\mathrm{Hilb}_q(R_\mu)$!

Returning to monomial bases of R_{1^n}

Question

Find a description of a monomial basis of R_μ that is

- a subset of the Artin basis/Descent basis,
- compatible with

$$\mathrm{Frob_q}(R_\mu) = \sum_{\substack{T \in \mathrm{SYT}_n \\ \mathrm{ctype}(T^t) \trianglerighteq \mu}} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T^t)}$$

$$\mathrm{Hilb_q}(R_\mu) = \sum_{\substack{w \in S_n \\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu}} q^{\mathrm{charge}(w)}.$$

.

Charge monomial basis of R_μ

Theorem (H. 2024+)

The set

$$\{\mathbf{x}^{\mathrm{c}(w)} \mid w \in S_n,\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu\}$$

is a monomial basis of R_μ.

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Remark

This basis coincides with the basis given by Carlsson-Chou.

Example: Charge monomial basis of $R_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}}$

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We know $\mathrm{ctype}\left(\begin{array}{|c|c}\hline 4 & \\ \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\right) = \begin{array}{|c|c}\hline & \\ \hline & \\ \hline & \multicolumn{1}{c|}{} \\ \hline \end{array}$.

Example: Charge monomial basis of $R_{(2,1,1)}$

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There are three different $w \in S_n$ such that $P(w) = \left(\begin{array}{|c|c}\hline 4 & \\ \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\right)^t = \begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline\end{array}$:

$$w = 2134$$
$$w = 2314$$
$$w = 2341$$

Example: Charge monomial basis of $R_{\ydiagram{1,1,2}}$

Theorem (H. 2024+)

The set $\{\mathbf{x}^{\mathrm{c}(w)} \mid w \in S_n,\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu\}$ is a monomial basis of R_μ.

We know $\mathrm{ctype}\left(\begin{ytableau} 4 \\ 3 \\ 1 & 2 \end{ytableau}\right) = \ydiagram{1,1,2}$.

There are three different $w \in S_n$ such that $P(w) = \left(\begin{ytableau} 4 \\ 3 \\ 1 & 2 \end{ytableau}\right)^t = \begin{ytableau} 2 \\ 1 & 3 & 4 \end{ytableau}$:

$$w = 2134 \to \mathrm{c}(w) = 0012$$
$$w = 2314$$
$$w = 2341$$

Example: Charge monomial basis of $R_{\ydiagram{1,1,2}}$

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$$w = 2134 \to \mathrm{c}(w) = 0012$$
$$w = 2314 \to \mathrm{c}(w) = 0102$$
$$w = 2341 \to \mathrm{c}(w) = 0120$$

Example: Charge monomial basis of $R_{\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline & \\ \hline\end{array}}$

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$$w = 2134 \to \mathrm{c}(w) = 0012$$
$$w = 2314 \to \mathrm{c}(w) = 0102$$
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Thus $x_3x_4^2, x_2x_4^2, x_2x_3^2$ are all in $\{\mathbf{x}^{\mathrm{c}(w)} \mid w \in S_n,\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu\}$

Example: Charge monomial basis of $R_{⊞}$

S	$\{w \mid P(w) = S\}$	$\{x^{c(w)} \mid P(w) = S\}$
2 1 3 4	$\{2134,\ 2314,\ 2341\}$	$\{x_3x_4^2,\ x_2x_4^2,\ x_2x_3^2\}$
2 4 1 3	$\{2143\ ,\ 2413\}$	$\{x_3x_4,\ x_2x_4\}$
4 2 1 3	$\{4213,\ 4231,\ 2431\}$	$\{x_1x_4,\ x_1x_3,\ x_2x_3\}$
3 2 1 4	$\{3214,\ 3241\ ,\ 3421\}$	$\{x_4,\ x_3,\ x_2\}$
4 3 2 1	$\{4321\}$	$\{1\}$

Charge monomial basis of R_μ

Theorem (H. 2024+)

The set

$$\{\mathbf{x}^{\mathrm{c}(w)} \mid w \in S_n,\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu\}$$

is a monomial basis of R_μ.

Why is this basis nice?

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- It is a subset of the descent basis of R_{1^n}. (charge $\leftrightarrow$ maj)

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Why is this basis nice?

- It is a subset of the descent basis of R_{1^n}. (charge $\leftrightarrow$ maj)
- It is compatible with

$$\mathrm{Hilb}_{\mathrm{q}}(R_\mu) = \sum_{\substack{w \in S_n \\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu}} q^{\mathrm{charge}(w)}.$$

Connections to $\mathrm{Frob_q}(R_\mu)$

For $\mathbb{C}S_n$-module V, $\mathrm{Frob_q}(V)$ is determined by

$$\mathrm{Hilb_q}(N_\gamma V) = \langle e_\gamma, \mathrm{Frob_q}(V)\rangle$$

for all $\gamma \vdash n$, where

$$N_\gamma = \sum_{\sigma \in S_\gamma} \mathrm{sgn}(\sigma)\sigma.$$

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$$N_\gamma = \sum_{\sigma \in S_\gamma} \mathrm{sgn}(\sigma)\sigma.$$

Proposition (H. 2024+)

Let $\mu, \gamma \vdash n$. The set

$$\{N_\gamma \mathbf{x}^{\mathrm{c}(w)} \mid w \in S_n,\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu,\ \mathrm{Des}(w) \subset \{\gamma_1, \gamma_1+\gamma_2, \ldots, \gamma_1+\cdots+\gamma_{l-1}\}\}.$$

is a basis of $N_\gamma R_\mu$ where $\mathrm{Des}(w) = \{i \mid w_i > w_{i+1}\}$.

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Corollary (H. 2024+)

We have

$$\mathrm{Frob}_q(R_\mu) = \tilde{H}_\mu[X; q].$$

Connections to $\mathrm{Frob}_{\mathrm{q}}(R_\mu)$

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is a basis of $N_\gamma R_\mu$ where $\mathrm{Des}(w) = \{i \mid w_i > w_{i+1}\}$.

$$\mathrm{Hilb}_{\mathrm{q}}(N_\gamma R_\mu) = \sum_{\substack{w \in S_n \\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu \\ \mathrm{Des}(w) \subset \{\gamma_1, \gamma_1+\gamma_2, \ldots, \gamma_1+\cdots+\gamma_{l-1}\}}} q^{\mathrm{charge}(w)} = \langle e_\gamma,\ \tilde{H}_\mu[X;q] \rangle.$$

Example: $\mu = (2,1,1)$, $\gamma = (2,2)$

There are 5 SYT P that satisfy $\text{ctype}(P^t) \trianglerighteq (2,1,1)$:

$$\begin{array}{|c|cc}\hline 2 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}\ ,\quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}\ ,\quad \begin{array}{|c|c}\hline 4 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\ ,\quad \begin{array}{|c|c}\hline 3 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array}\ ,\quad \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}.$$

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$$\begin{ytableau} 2 \\ 1 & 3 & 4 \end{ytableau}\,, \quad \begin{ytableau} 2 & 4 \\ 1 & 3 \end{ytableau}\,, \quad \begin{ytableau} 4 \\ 2 \\ 1 & 3 \end{ytableau}\,, \quad \begin{ytableau} 3 \\ 2 \\ 1 & 4 \end{ytableau}\,, \quad \begin{ytableau} 4 \\ 3 \\ 2 \\ 1 \end{ytableau}.$$

Note that $\mathrm{Des}(w) = \mathrm{Des}(Q(w))$. There are 3 SYT Q such that $\mathrm{Des}(Q) \subset \{2\} = \{\gamma_1\}$:

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We have two pairs (P,Q) where P,Q are the same shape

$$\left(\begin{array}{|c|cc}\hline 2 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}, \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \right) \leftrightarrow w = 2314,$$

$$\left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \right) \leftrightarrow w = 2413.$$

Example: $\mu = (2,1,1)$, $\gamma = (2,2)$

We have two pairs (P,Q) where P,Q are the same shape

$$\left(\begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline\end{array}, \begin{array}{|c|cc}\hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline\end{array}\right) \leftrightarrow w = 2314,$$

$$\left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}\right) \leftrightarrow w = 2413.$$

We compute the corresponding charge monomials:

$$w = 2314 \to \mathbf{x}^{c(w)} = x_2 x_4^2$$
$$w = 2413 \to \mathbf{x}^{c(w)} = x_2 x_4.$$

The basis of $N_\gamma R_\mu$ is

$$\{N_\gamma(x_2x_4^2), N_\gamma(x_2x_4)\} = \{x_2x_4^2 - x_1x_4^2 - x_2x_3^2 + x_1x_3^2, x_2x_4 - x_1x_4 - x_2x_3 + x_1x_3\}.$$

Thank you!