Modules for k-Atoms and a Combinatorial Formula

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Skew-linked partitions

Definition

Partitions λ and μ are *skew linked*, written

$$\lambda \xrightarrow{\theta} \mu$$

if there exists a skew diagram θ with the same row lengths (in order) as λ and the same column lengths as μ.

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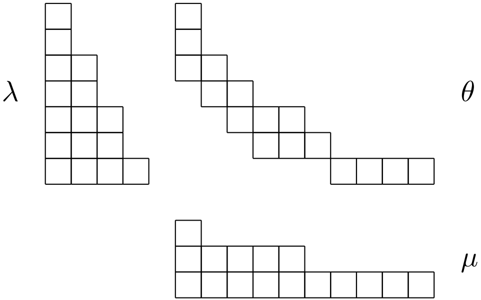

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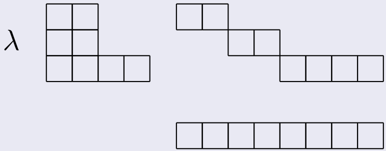

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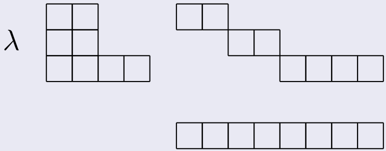

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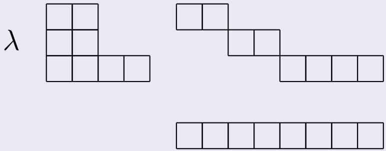

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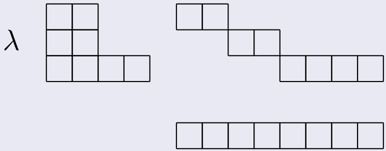

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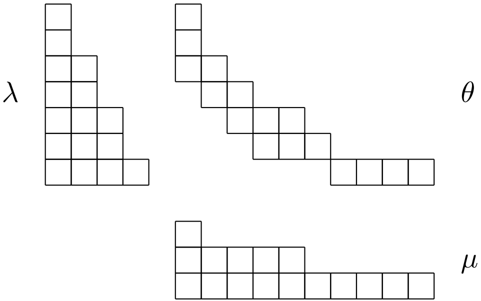
- Transpose symmetry: $\lambda \xrightarrow{\theta} \mu$ if and only if $\mu' \xrightarrow{\theta'} \lambda'$
- The two partitions λ and μ determine θ (and conversely, of course).

The "k-atom" case

Let κ be a $(k+1)$-*core* (no hook-length $= k+1$), and let θ be the set of boxes in κ with hook-length at most k.

Example ($k = 4$):



Then θ skew-links a k-bounded partition λ to the transpose of its Lapointe-Morse *k-conjugate*:

$$\lambda \xrightarrow{\theta} \mu = (\lambda^{[k]})'$$

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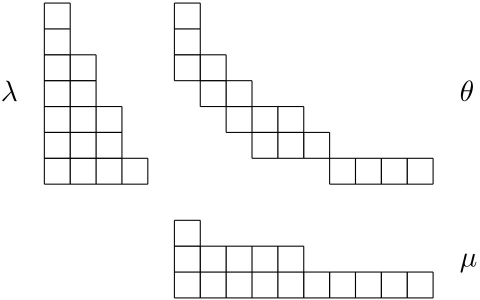
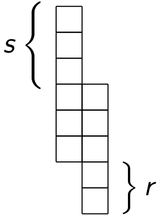


These cases, along with the $\mu = (n)$ cases, are the key examples!

Decomposing a skew-linking shape θ into row chains

Consecutive columns in a skew-linking shape θ always have $r \leq s$, with r and s as shown at right. Hence we can match the beginning of each row to the end of some higher row.



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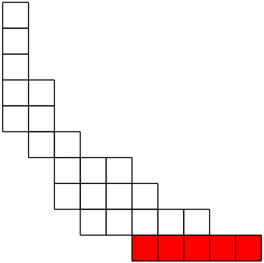

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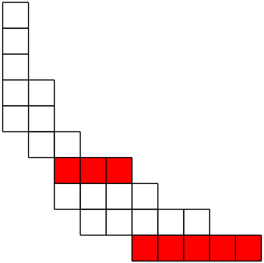

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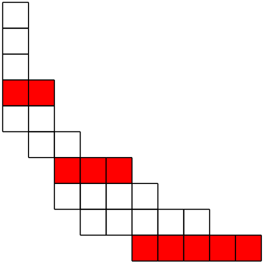

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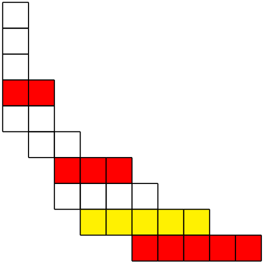

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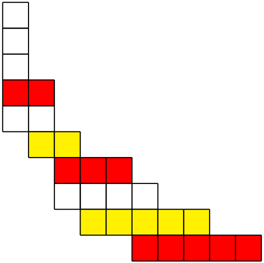

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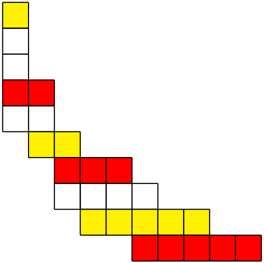

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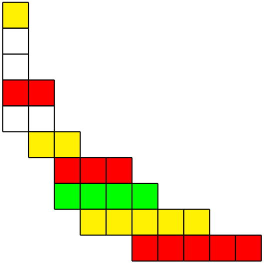

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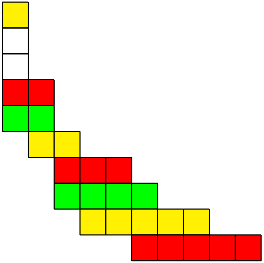

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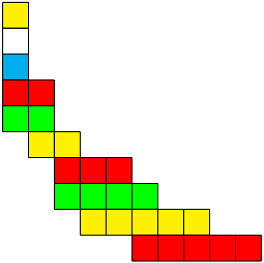

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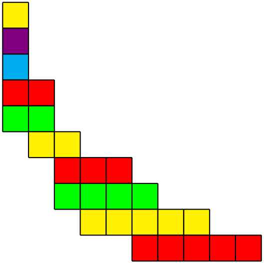

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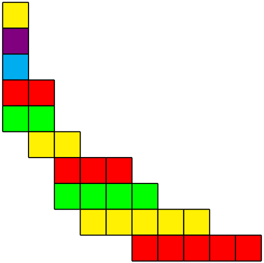
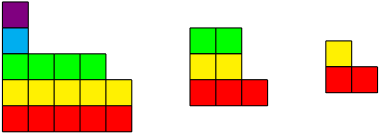


Now we group the rows into partitions, according to how far each row is from the end of its chain.



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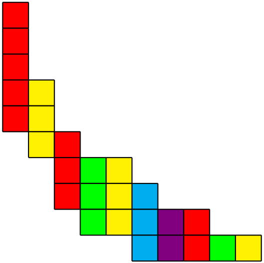
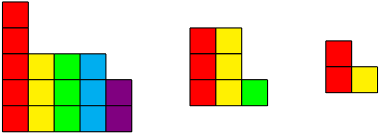


A remarkable fact is that doing it by columns leads to the same tuple of partitions.



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Some other (easy) facts

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- We have diagram containments

  $$\nu^{(1)} \supseteq \nu^{(2)} \supseteq \cdots \supseteq \nu^{(r)}.$$

  In particular,

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- The statistic

  $$n(\gamma) \underset{\text{def}}{=} \sum_i (i-1)\,\gamma_i = \sum_i (i-1)\,|\nu^{(i)}|$$

  is equal to the number of "missing boxes," $|\beta|$, where $\theta = \alpha/\beta$.

How to construct small $\mathbb{C}[\mathbf{x}] * S_n$ modules

Note: "$\mathbb{C}[\mathbf{x}] * S_n$ module" $=$ "$\mathbb{C}[x_1, \ldots, x_n]$ module with S_n action."

Motivation: How to construct irreducible S_n-modules.

Let $V = \varepsilon \uparrow_{S_{\lambda'}}^{S_n}$ be the S_n module induced from the sign representation of the Young subgroup $S_{\lambda'}$.

Let $W = 1 \uparrow_{S_\lambda}^{S_n}$ be induced from the trivial representation of the Young subgroup S_λ.

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The irreducible V_λ is the image of the essentially unique homomorphism

$$V \underset{\phi}{\rightarrow} W.$$

This uniquely characterizes V_λ as

1. generated by an (essentially unique) $S_{\lambda'}$-antisymmetric element, and
2. *co-generated* by an (essentially unique) S_λ-invariant linear functional.

Question

Which $\mathbb{C}[\mathbf{x}] * S_n$ modules can be characterized in a similar fashion?

Let $V = \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]$, the free $\mathbb{C}[\mathbf{x}]$ module on our previously considered induced S_n module.

Let $W = \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*$, a co-free $\mathbb{C}[\mathbf{x}]$ module on an induced S_n module, but we may have $\mu \neq \lambda$.

Let d be the smallest degree such that there is a non-zero S_n-module homomorphism

$$\psi\colon \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_d \to 1 \uparrow_{S_\mu}^{S_n} .$$

Suppose further that λ and μ are such that ψ is essentially unique.

With $V = \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]$ and $W = \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*$, let d = smallest degree such that there is a non-zero homomorphism

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Proposition

With the above hypotheses, there is an essentially unique $\mathbb{C}[\mathbf{x}] * S_n$ homomorphism, homogeneous of degree zero

$$\phi\colon V \to W[-d].$$

Its image $M_{\lambda,\mu}$ is a graded $\mathbb{C}[\mathbf{x}] * S_n$ module uniquely characterized as

1. *generated by an (essentially unique) $S_{\lambda'}$-antisymmetric element (in degree 0), and*
2. *co-generated by an (essentially unique) S_μ-invariant linear functional (on the top degree, which is equal to d).*

Main theorem

Theorem (C)

1. *The necessary and sufficient condition for the hypotheses of the preceding proposition to hold is that λ be skew-linked to μ.*
2. *In that case, the degree $d =$ (top degree of $M_{\lambda,\mu}$) is equal to $n(\gamma) = |\beta|$, where the skew diagram linking λ to μ is $\theta = \alpha/\beta$.*
3. *Moreover, the degree zero and top degree components of $M_{\lambda,\mu}$ are irreducible S_n modules isomorphic to V_λ and V_μ, respectively.*

Transpose symmetry

Recall $V = \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]$ and $W = \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*$. Suppose we dualize the essentially unique $\phi\colon V \to W[-d]$, then tensor with ε, the sign representation of S_n. The result is a nonzero homomorphism $\sigma\colon \left(\varepsilon \uparrow_{S_{(\mu')'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}] \to \left(\left(1 \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*\right)[-d]$. Since $\mu' \xrightarrow{\theta'} \lambda'$ with the same d, the image of σ is $M_{\mu',\lambda'}$.

Thus we obtain $M_{\mu',\lambda'}$ from $M_{\lambda,\mu}$ by dualizing and tensoring with ε. Dualizing reverses the degree, while tensoring with ε changes each copy of V_α to $V_{\alpha'}$.

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Proposition

Suppose the graded Frobenius characteristic of $M_{\lambda,\mu}$ is $\sum_\alpha f_\alpha(t) S_\alpha(z)$, where $f_\alpha(t) \in \mathbb{N}[t]$. Then the graded Frobenius characteristic of $M_{\mu',\lambda'}$ is $t^d \sum_\alpha f_\alpha(t^{-1}) S_{\alpha'}(z)$.

Special cases

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$$H_\lambda(z;t) = \sum_\kappa K_{\kappa,\lambda}(t) S_\kappa(z).$$

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Remark: Garsia and Procesi prove the character formula directly from the structure of the module. Conceivably, we might determine the character of a general $M_{\lambda,\mu}$ by similarly elementary means.

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Conjecture

If λ is k-bounded and $\mu = (\lambda^{[k]})'$ is the transpose of its k-conjugate, then the graded Frobenius characteristic of $M_{\lambda,\mu}$ is equal to the k-atom $A_\lambda^{(k)}(z;t)$ of Lascoux, Lapointe and Morse.

Bits of the proof of the uniqueness theorem

Goal: characterize λ, μ such that the space

$$\mathrm{Hom}_{S_n}\big(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_d,\ 1 \uparrow_{S_\mu}^{S_n}\big)$$

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Lemma

The desired degree d is the minimum of

$$\sum_{i,j} \binom{a_{i,j}}{2}$$

over all non-negative integer matrices A with row sums μ and column sums λ'.

The desired dimension-one condition holds if and only if the minimizing matrix A is unique.

Proposition (C)

A matrix A with specified, weakly decreasing row and column sums uniquely minimizes $\sum_{i,j}\binom{a_{i,j}}{2}$ iff it satisfies the following condition:

For every 2×2 minor $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of A, we have

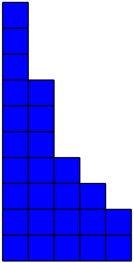
$$\begin{aligned} (a+d)-(b+c) &\leq 1 \quad \text{if } a,d>0 \\ (b+c)-(a+d) &\leq 1 \quad \text{if } b,c>0 \end{aligned}$$

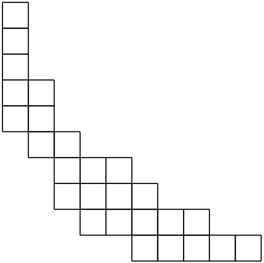
Such a matrix A must have the entries $a_{i,j}$ weakly decrease along rows and columns, i.e., A is a plane partition.

Moreover, there exists such a matrix A with column sums λ' and row sums μ if and only λ is skew-linked to μ, in which case A is given by the plane partition with layers $(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(r)})$.

Example:

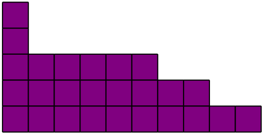
λ



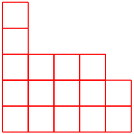


$$A = \begin{pmatrix} 3 & 3 & 2 & 1 & 1 \\ 3 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} 10 \\ 8 \\ 6 \\ 1 \\ 1 \end{matrix}$$

$$\begin{matrix} 10 & 7 & 4 & 3 & 2 \end{matrix}$$



μ



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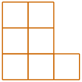

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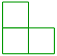

$\nu^{(3)}$

For the extra conclusion of the uniqueness theorem, that $M_{\lambda,\mu}$ is generated by V_λ and co-generated by V_μ, we must also prove that

$$\langle \chi^\lambda \otimes \mathrm{ch}(\mathbb{C}[\mathbf{x}]_d),\ \chi^\mu \rangle \ \neq\ 0.$$

This follows from

1. $d = \sum_i (i-1)\,|\nu^{(i)}|$, and
2. the Littlewood-Richardson coefficients $c^\lambda_{\nu^{(1)},\dots,\nu^{(r)}}$ and $c^\mu_{\nu^{(1)},\dots,\nu^{(r)}}$ are both non-zero.

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In fact, this implies that

$$\langle \chi^\lambda \otimes \chi^\gamma,\, \chi^\mu \rangle = 1.$$

Note that λ', μ and γ are the three projections of the plane partition given by the matrix A.

Charge

Definition

Given a word of partition weight, label its letters in the following way.

- Let $\ell = 0$.
- Starting from the end of the word and scanning backward, give label ℓ to the first 1, the first 2 following this 1, the first 3 following this 2, and so on.
- When the next letter (say p) is not found, start again at the end of the word and increment ℓ by 1. Give label ℓ to the first p, the first $p+1$ following this p, and so on.
- Keep scanning, incrementing ℓ as necessary, until one of each letter has been labelled.
- Repeat the above procedure on the unlabelled letters, each time resetting $\ell = 0$, until all letters have been labelled.

Definition (continued from previous slide)

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Example: $T = \begin{array}{|c|c|ccc} \hline 3 & 6 \\ \hline 2 & 2 \\ \hline 1 & 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array}, w = 362211345.$

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Example: $T = \begin{array}{|c|c|ccc}\hline 3 & 6 & \multicolumn{3}{c}{} \\ \hline 2 & 2 & \multicolumn{3}{c}{} \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$, $w = 362211345$.

The labelling passes through w are

$$3_0 6 2 2_0 1 1_0 3 4 5 \quad 3_0 6 2 2_0 1 1_0 3 4_1 5 \quad 3_0 6_2 2 2_0 1 1_0 3 4_1 5_2$$

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Charge $= 0 + 2 + 0 + 0 + 0 + 0 + 1 + 1 + 2 = 6.$

One step of catabolism

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- Let V be the tableau given by the portion of T above the first r rows.
- Denote by UV the skew tableau obtained by juxtaposing U to the northwest corner of V.
- Let $cat_r(T)$ be the unique tableau that is Knuth equivalent to UV.

Example:

$$r = 2$$

$$T = \begin{array}{|c|c|c|c|c|} \cline{1-1} 7 \\ \cline{1-3} 3 & 5 & 6 \\ \cline{1-4} 2 & 2 & 4 & 8 \\ \hline 1 & 1 & 1 & 3 & 4 \\ \hline \end{array} \qquad U = \begin{array}{|c|c|c|} \cline{1-2} 4 & 8 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 \\ \cline{2-3} \end{array} \qquad V = \begin{array}{|c|c|c|} \cline{1-1} 7 \\ \hline 3 & 5 & 6 \\ \hline \end{array}$$

$$UV = \begin{array}{cccccc} \cline{1-2} \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{8} & & & & \\ \cline{1-3} & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & & & \\ \cline{2-4} & & & \multicolumn{1}{|c|}{7} & & \\ \cline{4-6} & & & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\ \cline{4-6} \end{array} \equiv \begin{array}{|c|c|c|c|} \cline{1-1} 8 \\ \cline{1-3} 4 & 4 & 7 \\ \hline 3 & 3 & 5 & 6 \\ \hline \end{array} = cat_r(T)$$

Catabolism sequence

Definition

Let $r_1, r_2, \ldots, r_m$ be a sequence of positive integers. A tableau T is $r_1, \ldots, r_m$-*catabolizable* if there exists a sequence of tableaux $T_0 = T, T_1, \ldots, T_m$ such that T_{i-1} is r_i-catabolizable with $T_i = cat_{r_i}(T_{i-1})$ for $i = 1, \ldots, m$. Denote $cat_{r_1, r_2, \ldots, r_i}(T) = T_i$.

Example: T is $2, 2, 1$-catabolizable

$$T = \begin{array}{|c|c|c|c|c|} \hline 7 \\ \cline{1-1} 3 & 5 & 6 \\ \cline{1-3} 2 & 2 & 4 & 8 \\ \cline{1-4} 1 & 1 & 1 & 3 & 4 \\ \hline \end{array} \qquad cat_2(T) = \begin{array}{|c|c|c|c|} \hline 8 \\ \cline{1-1} 4 & 4 & 7 \\ \cline{1-3} 3 & 3 & 5 & 6 \\ \hline \end{array}$$

$$cat_{2,2}(T) = \begin{array}{|c|c|c|} \hline 7 \\ \cline{1-1} 5 & 6 & 8 \\ \hline \end{array} \qquad cat_{2,2,1}(T) = \begin{array}{|c|c|} \hline 8 \\ \cline{1-1} 6 & 7 \\ \hline \end{array}$$

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- Below all tableaux will have partition weight. Even in this case, a catabolism sequence needs not be a partition.
- Note that all tableaux are 1-catabolizable.
- If λ is a partition, then all (semistandard) tableaux of weight λ are $1^{\ell(\lambda)}$-catabolizable.
- On the other hand, the only $\ell(\lambda)$-catabolizable tableau of weight λ is the superstandard tableau, which is catabolizable with respect to every sequence.

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If a row starts at column 0, by convention we consider it to be chained on the left to row $\ell(\lambda) + 1$.

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Definition

Call a row-chaining scheme *monotone* if for $i < i'$, if row i is chained on the left to row j and row i' is chained on the left to row j', then either $j < j'$ or $j = j' = \ell(\lambda)+1$.

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Proposition

For each i, there exist constants b_i, d_i such that monotone row-chaining schemes can chain row i on the left to any row $j \in [i+b_i, i+d_i]$ but no other row.

1. Suppose row i does not start at column 0. Let rows $i-r,\ldots,i+s$ be the ones that begin at the same place as row i. Let rows $i',\ldots,i''$ be the ones that end at the same place as where row i begins. Then

$$b_i = i' + r - i, d_i = i'' - s - i.$$

2. If row i does start at column 0, then

$$b_i = d_i = \ell(\lambda) + 1 - i.$$

Tableau atoms

Definition

Let $\lambda \xrightarrow{\theta} \mu$ and define b_i, d_i as above. Define the tableau atom $\mathbb{A}_{\lambda,\mu}$ to be the set of tableaux of weight λ that are $r_1, \ldots, r_m$-catabolizable whenever

1. $r_1 + \ldots + r_m = \ell(\lambda)$, and
2. $r_{i+1} \leq d_{r_1+\ldots+r_i+1}$ for $i = 0, \ldots, m-1$.

Define $A_{\lambda,\mu}(z;t) = \sum_{T \in \mathbb{A}_{\lambda,\mu}} t^{charge(T)} S_{shape(T)}(z)$.

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Let θ^r be the result of removing the first r rows of θ. Notice $\lambda^r \xrightarrow{\theta^r} \mu^r$ for $\lambda^r = (\lambda_{r+1}, \lambda_{r+2}, \ldots)$ and some partition μ^r. Then $\mathbb{A}_{\lambda,\mu}$ is the set of tableaux T of weight λ such that for every $r = 1, 2, \ldots, d_1$,

1. T is r-catabolizable, and
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Conjecture

$A_{\lambda,\mu}(z;t)$ is the graded Frobenius characteristic of $M_{\lambda,\mu}$.

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Conjecture

If λ is k-bounded and $\mu = (\lambda^{[k]})'$ is the transpose of its k-conjugate, then $\mathbb{A}_{\lambda,\mu}$ coincides with the tableau atom $\mathbb{A}_{\lambda}^{(k)}$ of Lascoux, Lapointe and Morse.

Notable special cases

1. When $\lambda = \mu$, $d_1 = \ell(\lambda)$, so every $T \in \mathbb{A}_{\lambda,\lambda}$ is $\ell(\lambda)$-catabolizable. Thus $\mathbb{A}_{\lambda,\lambda}$ consists only of the superstandard tableau. It has charge 0, so $A_{\lambda,\lambda}(z;t) = S_\lambda(z)$ as required.

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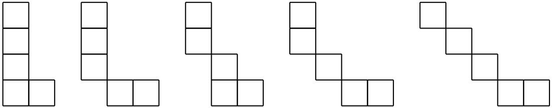
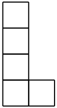
2. When $\mu = (n)$, $d_i = 1$ for all i, so $\mathbb{A}_{\lambda,(n)}$ consists of all $1^{\ell(\lambda)}$-catabolizable tableaux, *i.e.* *all* tableaux of weight λ. Thus

$$
\begin{aligned}
A_{\lambda,(n)}(z;t) &= \sum_{\kappa} \sum_{T \in SSYT(\kappa,\lambda)} t^{charge(T)} S_\kappa(z) \\
&= \sum_{\kappa} K_{\kappa,\lambda}(t) S_\kappa(z) \\
&= H_\lambda(z;t).
\end{aligned}
$$

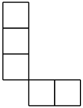
Examples: Let $\lambda = (2,1,1,1)$. The skew-linking shapes for λ are



 : $\begin{matrix} b_1 = 4 & b_2 = 3 & b_3 = 2 & b_4 = 1 \\ d_1 = 4 & d_2 = 3 & d_3 = 2 & d_4 = 1 \end{matrix}$

catabolism sequences = all compositions of 4

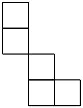
$$\mathbb{A}_{2111,2111} = \left\{ \begin{array}{|c|c} \hline 4 & \\ \hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \right\}, A_{2111,2111}(z;t) = t^0 S_{2111}(z)$$


: $b_1 = 1 \quad b_2 = 3 \quad b_3 = 2 \quad b_4 = 1$
$d_1 = 3 \quad d_2 = 3 \quad d_3 = 2 \quad d_4 = 1$

catabolism sequences = all compositions of 4 except (4)

$$\mathbb{A}_{2111,2111} = \left\{ \begin{array}{|c|cc} \hline 3 & & \\ \hline 2 & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} \\ \hline \end{array}, \begin{array}{|c|c} \hline 4 & \\ \hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array} \right\}$$

$$A_{2111,311}(z;t) = t^1 S_{311}(z) + t^0 S_{2111}(z)$$

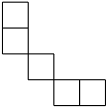
 :

$$b_1 = 2 \quad b_2 = 1 \quad b_3 = 2 \quad b_4 = 1$$
$$d_1 = 2 \quad d_2 = 2 \quad d_3 = 2 \quad d_4 = 1$$

catabolism sequences = 22, 121, 112, 1111

$$\mathbb{A}_{2111,32} = \left\{ \begin{array}{lll} 2 & 4 & \\ 1 & 1 & 3 \end{array}, \begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 1 & 4 \end{array}, \begin{array}{ll} 4 & \\ 3 & \\ 2 & \\ 1 & 1 \end{array} \right\}$$

$$A_{2111,32}(z;t) = t^2 S_{32}(z) + t^1 S_{311}(z) + t^0 S_{2111}(z)$$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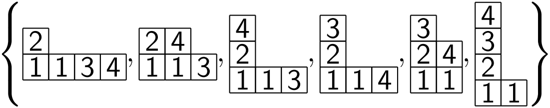
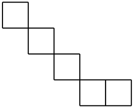
$\mathbb{A}_{2111,41} =$


$$A_{2111,41}(z;t) =$$
$$t^3 S_{41}(z) + t^2 S_{32}(z) + t^2 S_{311}(z) + t^1 S_{311}(z) + t^1 S_{221}(z) + t^0 S_{2111}(z)$$

 : $\begin{matrix} b_1 = 1 & b_2 = 1 & b_3 = 1 & b_4 = 1 \\ d_1 = 1 & d_2 = 1 & d_3 = 1 & d_4 = 1 \end{matrix}$

$$\text{catabolism sequences} = 1111$$

$$\mathbb{A}_{2111,5} = \cup_\kappa SSYT(\kappa, 2111)$$

$$A_{2111,5}(z;t) = H_{2111}(z;t)$$

Weaker catabolism conditions

Conjecture

Let θ^r, $\lambda^r \xrightarrow{\theta^r} \mu^r$ be as before. Let $i \in [b_1, d_1]$. Then $\mathbb{A}_{\lambda,\mu}$ is the set of tableaux T of weight λ such that for $r = 1$ and $r = i$,

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Remark: For $\mu = \lambda$ or $\mu = (n)$, we can check directly that the weaker catabolism condition suffices.
Suppose λ is k-bounded and its $(k+1)$-core induces $\lambda \xrightarrow{\theta} \mu$. Let $h = k + 1 - \lambda_1$ (the height of the first part of λ's k-split). Then

$$b_1 \leq h \leq d_1.$$

Thus the catabolism requirement for $\mathbb{A}_\lambda^{(k)}$ of Lascoux, Lapointe, and Morse is a special case.