Math H54 Honors Linear Algebra and Differential Equations Spring, 2004

Prof. Haiman

Homework 9 Solutions

4.4 Ex. 6. By symmetry, it's enough to verify that $||\mathbf{v}_1|| = 1$ and that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. For Ex. 9, part (a) is trivial, and part (b) is the sequence of dot products $\mathbf{y} \cdot \mathbf{v}_i$.

4.5 Ex. 4.

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}.$$

It would also be correct with the signs in the last column reversed.

4.5, Ex. 12. The matrix is orthogonal and equal to its own inverse (because it is symmetric).

4.4, Ex. 16.

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}; \quad R = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 1/\sqrt{2} \\ 0 & & 3/\sqrt{6} \end{bmatrix}$$

Problem A. Note that all matrices involved are invertible. Then $Q_1R_1 = Q_2R_2$ implies $Q_2^{-1}Q_1 = R_2R_1^{-1}$. Call this matrix D. By Theorem (4.63), D is orthogonal. Also, D is upper triangular, and hence, so is D^{-1}. But $D^{-1} = D^T$, so D is both upper- and lower-triangular. Therefore D is diagonal. Since its columns have norm equal to 1, the diagonal entries are ± 1. From the way we defined D and the fact that $D = D^{-1}$, we have $Q_2 = Q_1D$ and $R_2 = DR_1$.

Problem B. (a) Certainly Q has linearly independent columns, so the general formula for the matrix of the projection on CS(Q) gives $P = Q(Q^TQ)^{-1}Q^T$. But $Q^TQ = I$, so this simplifies to $P = QQ^T$.

(b) Both $Q_2Q_2^T$ and $Q_1Q_1^T$ are equal to the projection matrix P on the space $V =$ CS(Q_1) = CS(Q_2). Then

$$\begin{aligned} (Q_1^TQ_2)^T(Q_1^TQ_2) &= Q_2^TQ_1Q_1^TQ_2 \\ &= Q_2^TPQ_2 \\ &= Q_2^TQ_2 \\ &= I_n. \end{aligned}$$

Here the equation $PQ_2 = Q_2$ holds because every column $\mathbf{v}$ of Q_2 is in V and hence satisfies $P\mathbf{v} = \mathbf{v}$.