Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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Homework 7 Solutions

5.2 Ex. 8–10. Clearly $A_n \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^T = n \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^T$, so $\mathbf{v}_1 = \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^T$ is an eigenvector with eigenvalue $\lambda = n$. The vectors in Ex. 9 are in the nullspace of A_n, so they are eigenvectors with eigenvalue zero. They are linearly independent because the matrix whose rows are these vectors is in echelon form. The vector $\mathbf{v}_1$ is not in the span of the others, since it's not in the nullspace of A, so these n vectors are linearly independent. Hence A_n is diagonalizable with Λ having diagonal entries n (once) and 0 ($n-1$ times).

Now the characteristic polynomial of A is equal to that of Λ, namely $f(\lambda) = \lambda^{n-1}(\lambda - n)$. The matrix in the extra part of the problem is $I_n + A_n$, so its determinant is $\det(I_n + A_n) = (-1)^n \det(-I_n - A_n) = (-1)^n f(-1) = (-1)^n(-1)^{n-1}(-1-n) = n+1$.

5.2 Ex. 20. Characteristic polynomial is $(\lambda - 7)(\lambda + 3)^2$. Eigenvalues are $7, -3$, with -3 repeated twice. The vector

$$\begin{bmatrix}-2\\0\\1\end{bmatrix}$$

spans the $\lambda = 7$ eigenspace. The vectors

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}$$

span the $\lambda = 3$ eigenspace. You can verify that $7 - 3 - 3 = 1 = \operatorname{tr}(A)$ and $7(-3)^2 = 63 = \det(A)$.

5.2 Ex. 33. $\det(\lambda I_n - A) = \det((\lambda I_n - A)^T)$ by Theorem (2.19). Now since I_n is symmetric, $\det((\lambda I_n - A)^T) = \det(\lambda I_n - A^T)$, which is the characteristic polynomial of A^T.

5.3 Ex. 9. The eigenvalues are 2 and -3, and each eigenspace is 1-dimensional. So we can only find two linearly independent eigenvectors.

5.3 Ex. 26. Diagonalizable, with for instance

$$S = \begin{bmatrix}1 & 0 & 0 & 0\\4 & 1 & 0 & 0\\3 & 0 & 1 & 0\\2 & 0 & 0 & 1\end{bmatrix}, \quad \Lambda = \begin{bmatrix}6 & 0 & 0 & 0\\0 & 5 & 0 & 0\\0 & 0 & 5 & 0\\0 & 0 & 0 & 5\end{bmatrix}$$

5.3 Ex. 32. Since A and B have the same eigenvectors, we can diagonalize them with the same matrix S, so $A = S\Lambda S^{-1}$ and $B = S\Theta S^{-1}$, where Λ and Θ are diagonal. Diagonal matrices commute, so $AB = S\Lambda\Theta S^{-1} = S\Theta\Lambda S^{-1} = BA$.

Problem A. (a) $\phi(\mathbf{0}) = \mathbf{0}$, and if $\phi(\mathbf{v}) = \phi(\mathbf{w}) = \mathbf{0}$, then $\phi(\mathbf{v}+\mathbf{w}) = \mathbf{0}$ and $\phi(r\mathbf{v}) = \mathbf{0}$. This shows $\ker(\phi)$ contains zero and is closed under addition and scalar multiplication, so $\ker(\phi)$ is a subspace of V. Similarly $\phi(\mathbf{0}) = \mathbf{0}$ shows that $\mathbf{0} \in \operatorname{im}(\phi)$, and $\phi(\mathbf{v}) + \phi(\mathbf{w}) = \phi(\mathbf{v} + \mathbf{w})$, $r\phi(\mathbf{v}) = \phi(r\mathbf{v})$ shows that $\operatorname{im}(\phi)$ is closed under addition and scalar multiplication.

(b) We have $[\phi(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}}$. The left-hand side is zero if and only if $\mathbf{v} \in \ker(\phi)$, while the right-hand side is zero if and only if $[\mathbf{v}]_{\mathcal{B}} \in \mathrm{NS}(A)$.

(c) We have $\mathbf{w} \in \operatorname{im}(\phi)$ if and only if $\mathbf{w} = \phi(\mathbf{v})$ for some vector $\mathbf{v} \in V$, if and only if $[\mathbf{w}]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}}$, if and only if $[\mathbf{w}]_{\mathcal{C}} \in \mathrm{CS}(A)$.

(d) Let $\dim(V) = n$, $\dim(W) = n$, so A is an $m \times n$ matrix. Part (b) shows that $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ gives a linear isomorphism from $\ker(\phi)$ to $\mathrm{NS}(A)$, hence $\dim(\ker(\phi)) = \dim(\mathrm{NS}(A)) = n - \operatorname{rank} A$. Similarly, part (c) implies that $\dim(\operatorname{im}(\phi)) = \dim(\mathrm{CS}(A)) = \operatorname{rank} A$. Hence $\dim(V) = n = \dim(\ker(\phi)) + \dim(\operatorname{im}(\phi))$.

(e) T is linear, since the properties of derivatives easily give $T(f + g) = T(f) + T(g)$ and $T(rf) = rT(f)$ for constant r. The kernel of T is the set of solutions of the differential equation $f + f' = 0$. You can solve it by separation of variables (integrate $f'/f = -1$), giving the general solution $f(x) = Ae^{-x}$. This is only a polynomial if $A = 0$.

(f) By part (d), the image of $T\colon P_{<n} \to P_{<n}$ has dimension $n = \dim P_{<n}$, so T is onto, *i.e.*, every $g(x) \in P_{<n}$ is $T(f)$ for some f. Since this holds for every n, it follows that every polynomial $g(x)$ is equal to $f(x) + f'(x)$ for some polynomial $f(x)$. In fact, $f(x)$ is unique, because T has zero kernel in the space of polynomials.

Problem B. (a) $\Delta(f + g) = f(x + 1) + g(x + 1) - f(x) - g(x) = \Delta f + \Delta g$ and $\Delta(rf) = rf(x+1) - rf(x) = r\Delta(f)$.

(b) For $k > 0$,

$$\begin{aligned} \Delta C_k(x) &= \frac{(x+1)x(x-1)\cdots(x-k+2)}{k!} - \frac{x(x-1)\cdots(x-k+1)}{k!} \\ &= ((x+1) - (x-k+1))\frac{x(x-1)\cdots(x-k+2)}{k!} = \frac{x(x-1)\cdots(x-k+2)}{(k-1)!} = C_{k-1}(x). \end{aligned}$$

By definition, $\Delta C_0(x) = \Delta 1 = 0$. Hence the matrix A is $n \times n$ with entries 1 just above the diagonal and all other entries 0:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

(c) $\Delta 1 = 0$, $\Delta x = 1$, $\Delta x^2 = 2x + 1$, so

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$C_0 = 1$, $C_1 = x$, $C_2 = (x^2 - x)/2$, so

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

Now compute

$$XAX^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$