Homework 7 Solutions

5.2 Ex. 8–10. Clearly $A_n [1 \ 1 \ldots \ 1]^T = n [1 \ 1 \ldots \ 1]^T$, so $v_1 = [1 \ 1 \ldots \ 1]^T$ is an eigenvector with eigenvalue $\lambda = n$. The vectors in Ex. 9 are in the nullspace of $A_n$, so they are eigenvectors with eigenvalue zero. They are linearly independent because the matrix whose rows are these vectors is in echelon form. The vector $v_1$ is not in the span of the others, since it’s not in the nullspace of $A$, so these $n$ vectors are linearly independent. Hence $A_n$ is diagonalizable with $\Lambda$ having diagonal entries $n$ (once) and 0 ($n-1$ times).

Now the characteristic polynomial of $A$ is equal to that of $\Lambda$, namely $f(\lambda) = \lambda^{n-1}(\lambda - n)$. The matrix in the extra part of the problem is $I_n + A_n$, so its determinant is $\det(I_n + A_n) = (-1)^n \det(-I_n - A_n) = (-1)^n f(-1) = (-1)^n (-1)^{n-1}(-1 - n) = n + 1$.

5.2 Ex. 20. Characteristic polynomial is $(\lambda - 7)(\lambda + 3)^2$. Eigenvalues are 7, -3, with -3 repeated twice. The vector
\[
\begin{bmatrix}
-2 \\
0 \\
1
\end{bmatrix}
\]
spans the $\lambda = 7$ eigenspace. The vectors
\[
\begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]
span the $\lambda = 3$ eigenspace. You can verify that $7 - 3 - 3 = 1 = \text{tr}(A)$ and $7(-3)^2 = 63 = \det(A)$.

5.2 Ex. 33. $\det(\lambda I_n - A) = \det((\lambda I_n - A)^T)$ by Theorem (2.19). Now since $I_n$ is symmetric, $\det((\lambda I_n - A)^T) = \det(\lambda I_n - A^T)$, which is the characteristic polynomial of $A^T$.

5.3 Ex. 9. The eigenvalues are 2 and -3, and each eigenspace is 1-dimensional. So we can only find two linearly independent eigenvectors.

5.3 Ex. 26. Diagonalizable, with for instance
\[
S = \begin{bmatrix}
1 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 \\
2 & 0 & 0 & 1
\end{bmatrix}, \quad \Lambda = \begin{bmatrix}
6 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & 5
\end{bmatrix}
\]

5.3 Ex. 32. Since $A$ and $B$ have the same eigenvectors, we can diagonalize them with the same matrix $S$, so $A = S\Lambda S^{-1}$ and $B = S\Theta S^{-1}$, where $\Lambda$ and $\Theta$ are diagonal. Diagonal matrices commute, so $AB = S\Lambda\Theta S^{-1} = S\Theta\Lambda S^{-1} = BA$. 
Problem A. (a) $\phi(0) = 0$, and if $\phi(v) = \phi(w) = 0$, then $\phi(v+w) = 0$ and $\phi(rv) = 0$. This shows $\ker(\phi)$ contains zero and is closed under addition and scalar multiplication, so $\ker(\phi)$ is a subspace of $V$. Similarly $\phi(0) = 0$ shows that $0 \in \im(\phi)$, and $\phi(v)+\phi(w) = \phi(v+w)$, $r\phi(v) = \phi(rv)$ shows that $\im(\phi)$ is closed under addition and scalar multiplication.

(b) We have $[\phi(v)]_C = A[v]_B$. The left-hand side is zero if and only if $v \in \ker(\phi)$, while the right-hand side is zero if and only if $[v]_B \in \NS(A)$.

(c) We have $w \in \im(\phi)$ if and only if $w = \phi(v)$ for some vector $v \in V$, if and only if $[w]_C = A[v]_B$, if and only if $[w]_C \in \CS(A)$.

(d) Let $\dim(V) = n$, $\dim(W) = n$, so $A$ is an $m \times n$ matrix. Part (b) shows that $v \mapsto [v]_B$ gives a linear isomorphism from $\ker(\phi)$ to $\NS(A)$, hence $\dim(\ker(\phi)) = \dim(\NS(A)) = n - \rank A$. Similarly, part (c) implies that $\dim(\im(\phi)) = \dim(\CS(A)) = \rank A$. Hence $\dim(V) = n = \dim(\ker(\phi)) + \dim(\im(\phi))$.

(e) $T$ is linear, since the properties of derivatives easily give $T(f+g) = T(f) + T(g)$ and $T(rf) = rT(f)$ for constant $r$. The kernel of $T$ is the set of solutions of the differential equation $f + f' = 0$. You can solve it by separation of variables (integrate $f'/f = -1$), giving the general solution $f(x) = Ae^{-x}$. This is only a polynomial if $A = 0$.

(f) By part (d), the image of $T$: $P_{<n} \to P_{<n}$ has dimension $n = \dim P_{<n}$, so $T$ is onto, i.e., every $g(x) \in P_{<n}$ is $T(f)$ for some $f$. Since this holds for every $n$, it follows that every polynomial $g(x)$ is equal to $f(x) + f'(x)$ for some polynomial $f(x)$. In fact, $f(x)$ is unique, because $T$ has zero kernel in the space of polynomials.

Problem B. (a) $\Delta(f + g) = f(x+1) + g(x+1) - f(x) - g(x) = \Delta f + \Delta g$ and $\Delta(rf) = rf(x+1) - rf(x) = r\Delta(f)$.

(b) For $k > 0$,

$$\Delta C_k(x) = \frac{(x+1)x(x-1)\cdots(x-k+2)}{k!} - \frac{x(x-1)\cdots(x-k+1)}{k!} = \frac{(x+1)\cdots(x-k+2)}{k!} - \frac{x\cdots(x-k+2)}{(k-1)!} = C_{k-1}(x).$$

By definition, $\Delta C_0(x) = \Delta 1 = 0$. Hence the matrix $A$ is $n \times n$ with entries 1 just above the diagonal and all other entries 0:

$$A = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}.$$ 

(c) $\Delta 1 = 0$, $\Delta x = 1$, $\Delta x^2 = 2x + 1$, so

$$B = \begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{bmatrix}.$$
$C_0 = 1$, $C_1 = x$, $C_2 = (x^2 - x)/2$, so

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}.$$ 

Now compute

$$XAX^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}. $$