Homework 6 Solutions

3.7 Ex. 18. $\dim \mathrm{RS}(A) = 2$, with one possible basis

$$\begin{bmatrix} 1 & -2 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 7 & -15 & 4 \end{bmatrix}.$$

$\dim \mathrm{NS}(A) = 2$, with one possible basis

$$\begin{bmatrix} -2 \\ 15 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} -15 \\ -4 \\ 0 \\ 7 \end{bmatrix}.$$

3.8 Ex. 18.

$$\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}, \quad [\mathbf{x}]_C = \begin{bmatrix} -1 & 0 \end{bmatrix}.$$

3.8 Ex. 31. The transition matrix from B to D is QP, since $[\mathbf{x}]_D = Q[\mathbf{x}]_C = Q(P[\mathbf{x}]_B)$.

4.1 Ex. 16.

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4.1 Ex. 28: Not linear. **30, 36**: Linear.

Problem A. (a) $W_1 + W_2$ contains $\mathbf{0} = \mathbf{0} + \mathbf{0}$, and if $w_1 + w_2$, $w_1' + w_2'$ are elements of $W_1 + W_2$, then $w_1 + w_2 + w_1' + w_2' = (w_1 + w_1') + (w_2 + w_2') \in W_1 + W_2$ and $r(w_1 + w_2) = rw_1 + rw_2 \in W_1 + W_2$. (b) Clearly, $\mathbf{0} \in W_1 \cap W_2$. If $v, w \in W_1 \cap W_2$, then $v + w \in W_1 \cap W_2$ since both W_1 and W_2 are subspaces; similarly $rv \in W_1 \cap W_2$.

Problem B. (a) Row operations don't change the row-space, so they don't change $\mathrm{rank}(A) = \dim(\mathrm{RS}(A))$. Column operations don't change the column space, so they don't change $\mathrm{rank}(A) = \dim(\mathrm{CS}(A))$.

(b) Using row operations we can first get a matrix U in row-echelon form. If the rank is r, then U has r pivots, and its column space consists of all vectors of the form

$$\begin{bmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Using column operations we can replace U by any matrix with the same size and column space; so we can reach the specified form.

Problem C. (a) Let B be a basis of W. Then $\dim(W) = |B|$, and $|B| \le \dim(V)$ because B is a linearly independent subset of V.

(b) If $\dim(W) = \dim(V)$, then B is a linearly independent set of size $\dim(V)$, and therefore a basis of V. Hence $V = \operatorname{span}(B) = W$.

Problem D. (a) $\mathrm{CS}(B) \subseteq \mathrm{NS}(A)$ means that $A\mathbf{x} = 0$ for every column $\mathbf{x}$ of B. That is equivalent to $AB = 0$.

(b) Since $\mathrm{CS}(B) = \mathrm{NS}(A)$, we have $AB = \mathbf{0}$ by part (a). Taking transposes, we have $B^T A^T = \mathbf{0}$, which implies $\mathrm{CS}(A^T) \subseteq \mathrm{NS}(B^T)$ by part (a) again. Say A has n columns and B has n rows; note that these are the same n. Then $\dim \mathrm{CS}(A^T) = \dim \mathrm{RS}(A) = \operatorname{rank}(A) = n - \dim \mathrm{NS}(A) = n - \operatorname{rank}(B)$ by the hypothesis $\mathrm{CS}(B) = \mathrm{NS}(A)$. In turn, $n - \operatorname{rank}(B) = \dim \mathrm{NS}(B^T)$. So we have shown that $\mathrm{CS}(A^T) \subseteq \mathrm{NS}(B^T)$ and that these two spaces have the same dimension. Then $\mathrm{CS}(A^T) = \mathrm{NS}(B^T)$ by Problem B part (b).

(c) For each subspace V or W, take A the transpose of the matrix with columns the given vectors, so $\mathrm{CS}(A^T)$ is the given subspace. Compute a basis for $\mathrm{NS}(A)$ and make it the columns of a matrix B. By part (b), the desired subspace is the nullspace of B^T. Carrying out this computation gives

$$V = \mathrm{NS}(X),\ X = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{bmatrix}; \quad W = \mathrm{NS}(Y),\ Y = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Now row-reduce the matrix

$$\begin{bmatrix} X \\ Y \end{bmatrix}$$

and compute a basis of its nullspace to be

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$