Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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Homework 5 Solutions

3.3 Ex. 18, 22, 24, 26, 28, 30. Ex. 18 is not a vector space because the property $1\mathbf{u} = \mathbf{u}$ fails. Ex. 24 is not a vector space because the property $r(\mathbf{u}+\mathbf{v}) = r\mathbf{u}+r\mathbf{v}$ fails. Ex. 26 is not a vector space because there is no additive identity element $\mathbf{0}$ (among other reasons). Exx. 22, 28 and 30 are vector spaces. In Exx. 22 and 28 they are subspaces of familiar spaces. In Ex. 30, the change of variables $x = e^u$ makes the operations agree with the usual ones on $u \in \mathbb{R}^1$.

3.4 Ex. 2, 4, 6, 17, 20. In Exx. 2, 17, and 20, S is a subspace of V. (In Ex. 20, the a, b, c in the definition of S should have been other letters, such as r, s, t. As it stands, a and b are used in two conflicting senses.) Ex. 4: S is closed under addition but not scalar multiplication. Ex. 6: S is not closed under either addition or scalar multiplication.

3.4 Ex. 48. The set of polynomials $ax + bx^2$ for all real numbers a and b.

Problem A.

$$\begin{bmatrix} -9 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

or any linear combinations of these that span the same subspace of $\mathbb{R}^5$.

Problem B. Suppose $\mathcal{B}$ is linearly dependent, so we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0$$

for some scalars c_i not all zero. Choose an index j such that $c_j \neq 0$. Then dividing the above equation by c_j and subtracting the terms for $i \neq j$ from both sides gives

$$\mathbf{v}_j = -\sum_{i \neq j} \frac{c_i}{c_j}\mathbf{v}_i.$$

Let $\mathcal{B}' = \{\mathbf{v}_i : i \neq j\}$. The preceding equation shows that $\mathbf{v}_j$ is in the span of $\mathcal{B}'$, and hence $\text{span}(\mathcal{B}') = \text{span}(\mathcal{B}) = V$. Thus a proper subset of $\mathcal{B}$ spans V.

It follows that if no proper subset of $\mathcal{B}$ spans V, then $\mathcal{B}$ is linearly independent.