Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
Prof. Haiman

Homework 4 Solutions

2.3 Ex. 26. $3x(x-1) - x(x+1) = 2x^2 - 4x$.

2.3 Ex. 28. -6

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$$\operatorname{adj}(A) = \begin{bmatrix} 0 & -16 & -16 \\ -14 & 1 & 8 \\ 0 & 40 & -16 \end{bmatrix}; \quad A^{-1} = \frac{-1}{112} \begin{bmatrix} 0 & -16 & -16 \\ -14 & 1 & 8 \\ 0 & 40 & -16 \end{bmatrix}.$$

Problem A. (a) Clearly $\det(A)$ is a polynomial in w, x, y, z. Since each term contains one factor from each column, it is a product of one variable to the first power, another squared, and a third cubed, for a total degree of 6.

(b) Setting two variables equal makes two rows of the matrix equal, hence makes the determinant become zero.

(c) The polynomial $d(w,x,y,z) = (x-w)(y-w)(z-w)(y-x)(z-x)(z-y)$ is a product of six linear factors, so all its terms have degree 6. Hence $f(w,x,y,z)$ is a multiple of $d(w,x,y,z)$ by a polynomial of degree zero, that is, a constant. The term xy^2z^3 occurs in $f(w,x,y,z)$ with coefficient $+1$ from the signed sum formula for the determinant. In $d(w,x,y,z)$ we can get this term in only one way, namely, by taking z from the factors $z-w$, $z-x$ and $z-y$, y from the factors $y-w$ and $y-x$, and x from the remaining factor $x-w$. Thus it also occurs with coefficient $+1$, so the constant c must be equal to 1.

Problem B. By cofactor expansion on the first row, we get $\det(A_n) = -\det(X)$, where X has the block form

$$X = \begin{bmatrix} 1 & 0 \\ 0 & A_{n-2} \end{bmatrix}.$$

Using cofactor expansion on the first row of X, we get $\det(X) = \det(A_{n-2})$. Thus $\det(A_n) = -\det(A_{n-2})$. Starting with $\det(A_1) = \det(\begin{bmatrix}0\end{bmatrix}) = 0$, it follows that $\det(A_n) = 0$ for all odd n. Starting with $\det(A_2) = \det(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]) = -1$, we get in turn $\det(A_4) = 1$, $\det(A_6) = -1$, and so on, the general rule being $\det(A_n) = (-1)^{n/2}$.