Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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Homework 3 Solutions

1.5 Ex. 30.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & -4 \\ 0 & 5 & 10 \\ 0 & 0 & -5 \end{bmatrix}$$

1.5 Ex. 36.

$$\mathbf{x} = \begin{bmatrix} -2 \\ -4 \\ -1 \\ 2 \end{bmatrix}$$

1.5 Ex. 39, 43. For 39, if $L = U$, this matrix is both upper and lower triangular, hence diagonal. Moreover, since it is lower *unit* triangular, it must be the identity matrix. For 43, suppose $LU = L'U'$. We assume A is invertible (this really should have been stated in the problem), so L, L', U and U' are also invertible (the L's are in any case). Then $(L')^{-1}L = U'(U^{-1})$, the matrix on the left-hand side is lower unit triangular, and the matrix on the right-hand side is upper-triangular. By Ex. 39, both matrices are the identity, so $L = L'$ and $U = U'$.

1.6 Ex. 20. (a) $(B + B^T)^T = B^T + (B^T)^T = B + B^T$. (b) $(B - B^T)^T = B^T - (B^T)^T = -B + B^T = -(B - B^T)$. **Ex. 22** $(BB^T)^T = (B^T)^T B^T = BB^T$; $(B^T B)^T = B^T (B^T)^T = B^T B$. Note that both products make sense since B^T is $n \times m$ if B is $m \times n$.

2.1 Ex. 20 54

2.2 Ex. 25 Since $-A = (-I_n)A$, and $\det(-I_n) = (-1)^n$, we have $\det(-A) = (-1)^n \det(A)$. Thus $\det(-A) = \det(A)$ if and only if n is even, or A is singular.

Problem A. "Since we can now calculate $\det A$ by row reduction for any square matrix A, we know that a determinant function exists." Wrong. What this shows is that a determinant function is unique if it exists.

"On the other hand, a matrix can be row reduced in several ways. Perhaps these different row reductions will give different determinants." It is true that we have not yet shown that different row reductions always give the same answer. However, in order to prove this, we need to show that a function with the properties in Definition (2.2) *exists*. Then, since we have shown that row reduction is a valid way to compute any such function, different row reductions must give the same answer. So the thing we need still to prove is that a determinant function exists.

The error continues in Section 2.3. Theorem 2.27 is not stated correctly. Instead it should read "For any square matrix A, if we define $\det A$ to be the sum of all signed elementary products from A, then $\det A$ defined in this way is a determinant function—in other words, it has the properties in Definition 2.2."

Corollary 2.28 should read "A determinant function exists."

I'll prove Theorem 2.27 (or rather, the correct version of it) in the lecture, since it's not done in the book.

Problem B. If A has integer entries, then $\det(A)$ is an integer. Unfortunately, this is not obvious from the method of calculation using row reduction. However, it is obvious from the formula for the determinant as a sum of signed elementary products, discussed in Section 2.2. Sorry about that—the homework got a bit ahead of the lecture here.

Anyway, granting the above fact, if A and A^{-1} are both integer matrices, then $\det(A)$ and $\det(A^{-1})$ are both integers, and $\det(A)\det(A^{-1}) = \det(I_n) = 1$. The only integer solutions of $pq = 1$ are $p = q = \pm 1$, so $\det(A) = \pm 1$.

Problem C. (a) Subtract $2\times$(row 2) from row 3, then subtract row 3 from row 4, and finally add $3\times$(row 1) to row 4. This gives the matrix W.

(b) Rearranging the rows of W in the order (row 2, row 3, row 1, row 4) gives a row-echelon matrix U. Hence $W = PU$, where P is the permutation that performs the inverse row rearrangement:

$$W = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 7 & -4 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

(c) We want $A = LW$, or $W = L^{-1}A$, so L is the matrix that does the inverse of the row operations in part (a), namely

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -3 & 0 & 1 & 1 \end{bmatrix} W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -3 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 7 & -4 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

(d) One simple possibility is to take

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then A is already lower unit triangular, so one $PA = LU$ factorization is simply

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To find a second factorization, switch the rows of A before row reducing. This gives

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$