Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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## Homework 10 Solutions

**Problem A**. Use the inner product $f \cdot g = \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx$. Take $g(x) = 1$. Then $\|g(x)\| = 1$. The desired inequality can be expressed as $|f(x) \cdot g(x)|^2 \leq \|f(x)\|^2$, which is a special case of Cauchy-Schwarz.

**Problem B**. (a) $x^k \cdot x^l = I_{k+l}$, where $I_m$ is given by the table.

(b) $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$.

(c) Obvious for $m = 0, 1$. For $m = 2, 3$, use trig identities to show $\cos 2x = 2\cos^2 x - 1$, $\cos 3x = 4\cos^3 x - 3\cos x$.

I didn't expect most students to give a proof for all $m$, but it can be done as follows. First, using the trig identity $\cos(n\theta) = 2\cos((n-1)\theta)\cos\theta - \cos((n-2)\theta)$ repeatedly, we see that some polynomial $C_n(x)$ of degree $n$ exists, such that $C_n(\cos\theta) = \cos(n\theta)$. From the same identity, it is not hard to work out that $C_n(x)$ has highest term $2^{n-1}x^n$, except for $C_0(x) = 1$.

To prove that $T_n(x) = C_n(x)$, we need to show that the polynomials $C_n(x)$ are orthogonal. Making a change of variables $x = \cos\theta$ in the integral that defines the inner product gives

$$C_m(x) \cdot C_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(m\theta)\cos(n\theta)\, d\theta.$$

This integral is zero for $m \neq n$, as we will see when we discuss Fourier series.