Math H54 — Honors Linear Algebra and Differential Equations — Spring, 2004

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Solutions to final exam review problems

1. The 6 matrices below form a basis:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

To prove that they form a basis it is enough to observe that every symmetric matrix is a linear combination of the above with unique coefficients:

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} = aA_1 + bA_2 + cA_3 + dA_4 + eA_5 + fA_6.$$

2. We can take $V = \mathrm{NS}(A)$, $W = \mathrm{NS}(B)$ for some 2×7 matrices A, B. Then $V \cap W = \mathrm{NS}(X)$, where

$$X = \begin{bmatrix} A \\ B \end{bmatrix}.$$

Since X is a 4×7 matrix, we have $\mathrm{rank}(X) \leq 4$, and hence $\dim(V \cap W) = 7 - \mathrm{rank}(X) \geq 3$.

3. Any vector $\mathbf{x} \in \mathrm{NS}(A) \cap \mathrm{RS}(A)$ satisfies $\mathbf{x}^T\mathbf{x} = \|\mathbf{x}\| = 0$, hence $\mathbf{x} = \mathbf{0}$.

4. (a) For every vector $\mathbf{v} \in \mathbb{R}^n$, we have $A(A\mathbf{v}) = A\mathbf{v}$, which shows that $A\mathbf{v}$ belongs to the $\lambda = 1$ eigenspace of A. Call this space E_1. We also have $A(\mathbf{v} - A\mathbf{v}) = 0$, so $\mathbf{v} - A\mathbf{v}$ belongs to the $\lambda = 0$ eigenspace (that is, the nullspace) of A. Call this space E_0. Since $\mathbf{v} = A\mathbf{v} + (\mathbf{v} - A\mathbf{v})$, we see that $E_0 + E_1 = \mathbb{R}^n$. This shows that eigenvectors of A span $\mathbb{R}^n$, and hence there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A, so A is diagonalizable.

(b) By part (a), A is diagonalizable, similar to a diagonal matrix Λ with diagonal entries 0 or 1. Then $\mathrm{tr}(A) = \mathrm{tr}(\Lambda)$ and $\mathrm{rank}(A) = \mathrm{rank}(\lambda)$. But $\mathrm{tr}(\Lambda) = \mathrm{rank}(\lambda)$, since either one is the number of diagonal entries that are equal to 1.

5. The condition on A implies that

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix}.$$

This shows that $[1\ 1\ \cdots\ 1]^T$ is an eigenvector with eigenvalue c.

If, instead, every column of A sums to c, then A^T has c as an eigenvalue by the preceding reasoning. But $\det(\lambda I - A) = \det(\lambda I - A^T)$, so A has the same eigenvalues as A^T. In this case, however, we are not given enough information to find a specific eigenvector associated with c.

6. $x(t) = 2t^{3/2} - t$.

7.

$$x(t) = \begin{cases} -\frac{1}{3} + \frac{1}{3}e^{3(t-1)}, & \text{for } 1 \le t < 2, \\ -\frac{1}{3}t + \frac{2}{9} + (\frac{1}{3}e^3 + \frac{1}{9})e^{3(t-2)}, & \text{for } t \ge 2. \end{cases}$$

8. $x(t) = (t-1)e^{4t} + e^{-2t}$.

9. $x(t) = e^{-\pi/2}e^t \cos 2t - e^{-\pi/2}e^t \sin 2t$.

10. $x(t) = (\frac{1}{3}t^4 + 2t + 1)e^{-2t}$.

11.

$$\mathbf{x}(t) = \begin{bmatrix} e^{2t} \\ -2e^{-t} + e^{2t} \\ e^{-t} - e^{2t} \end{bmatrix}.$$

12.

$$\mathbf{x}(t) = \begin{bmatrix} e^t + 3\cos t + \sin t \\ -3e^t - \cos t - 2\sin t \end{bmatrix}$$

13.

$$\mathbf{x}(t) = C_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} te^{2t} \\ e^{2t} + te^{2t} \end{bmatrix}.$$

14. It's clear that $x_1(t) = t^3$ is continuous and differentiable for all t, and you can check directly that it's a solution of the IVP. As for $x_2(t) = |t^3|$, it is also clearly continuous, and its derivative is $3t^2$ for $t > 0$ and $-3t^2$ for $t < 0$. At $t = 0$, its derivatives from the left and right are both equal to 0, so $x_2(t)$ is differentiable for all t, with derivative $3t^2$ for $t \ge 0$ and $-3t^2$ for $t \le 0$. Now you can again check directly that it's a solution of the IVP, by checking the cases $t \ge 0$ and $t \le 0$ separately.

This does not contradict the existence and uniqueness theorem because in standard form, the differential equation becomes

$$x'(t) - 3t^{-1}x(t) = 0.$$

Since the coefficient $-3t^{-1}$ is not defined at $t = 0$, the existence and uniqueness theorem only applies on the intervals $(0, \infty)$ and $(-\infty, 0)$ separately. The equation can (and does) have a non-unique solution on the whole real line.

15. Just check that they are solutions. For independence, compute the Wronskian

$$W = \det \begin{bmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{bmatrix} = 2t^3,$$

which is non-zero for $t \in (0, \infty)$.

16. $x(t) = C_1 + C_2 t + C_3 e^t \sin t + C_4 e^t \cos t + C_5 e^{-t} \sin t + C_6 e^{-t} \cos t$.