Solutions to review problems for Midterm 2

1. Let $V$ be the set of $4 \times 4$ matrices with all row- and column-sums equal to zero.
   (a) Show that $V$ is a subspace of $M_{4,4}$.
   (b) Find $\dim(V)$.

Solution. (a) $V$ is the solution set of a system of linear equations, hence a subspace.

   (b) Method 1: Set up coordinates on $M_{4,4}$. Then we have 8 equations (one for each row and column) in 16 variables. Working out the matrix and row-reducing it shows that one equation is redundant, so the solution space has dimension $16 - 7 = 9$.

   Method 2: The upper-left $3 \times 3$ block of $M$ can be chosen at will, and six of the equations then determine the remaining entries in the first 3 rows and first 3 columns. The entry in position $(4, 4)$ is then determined by both the equation for the last row and the equation for the last column, but both give the same result. This gives a linear isomorphism from $M_{3,3}$ to $V$, so $\dim(V) = 9$.

2. Consider the four functions in $C(\mathbb{R})$:
   
   $$f(x) = \cos^2 x, \quad g(x) = \sin^2 x, \quad h(x) = \cos 2x, \quad j(x) = \sin 2x.$$  

Are they linearly independent? Prove it if so; otherwise express one of them as a linear combination of the others.

Solution. $\cos 2x = \cos^2 x - \sin^2 x$

3. Prove that if $\text{CS}(A) = \text{NS}(A)$, then $A$ is a square matrix of even size.

Solution. Let $A$ be $m \times n$. Then $\text{CS}(A)$ is subspace of $\mathbb{R}^m$, while $\text{NS}(A)$ is a subspace of $\mathbb{R}^n$, so $m = n$. Now $\text{rank}(A) = \dim(\text{CS}(A)) = \dim(\text{NS}(A)) = n - \text{rank}(A)$, so $n = 2 \text{rank}(A)$, which shows that $n$ is even.

4. Chapter 3 Review Exercise 27.

Solution. See textbook.

5. (a) Find real numbers $w, x, y, z$ such that the characteristic polynomial of the matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
w & x & y & z
\end{bmatrix}
\]

is $(\lambda - 1)^4$.

(b) Is the resulting matrix diagonalizable? Why or why not?
Solution. (a)
\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 4 & -6 & 4
\end{bmatrix}
\]

(b) Not diagonalizable, since its eigenvalues are all equal to 1, but it is not the identity matrix.

6. Find a $3 \times 3$ matrix $X$ such that
\[
X^2 = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}.
\]

How many such matrices $X$ are there?

Solution. Diagonalize the given matrix as $S\Lambda S^{-1}$, where
\[
S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.
\]

To find $X$, replace $\Lambda$ by a square root, such as
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \text{giving} \quad X = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.
\]

The general form of a square root of $\Lambda$ is
\[
\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm 3 \end{bmatrix},
\]

for a total of 8 solutions.

7. Let $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ be the Fibonacci sequence. Suppose we extend the definition of $F_k$ to negative $k$ by requiring that $F_k = F_{k-1} + F_{k-2}$ hold for all $k$.

(a) Find a matrix $A$ such that
\[
\begin{bmatrix}
F_{-k} \\
F_{-(k-1)}
\end{bmatrix} = A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

for all $k$.

(b) How is $F_{-k}$ related to $F_k$?

Solution. (a) $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. (b) $F_{-k} = (-1)^{k-1}F_k$.

8. Let $V = \text{span}(e^x, xe^x, x^2e^x)$.
(a) Find the matrix of the linear transformation $D : V \to V$ defined by differentiation, with respect to the given basis of $V$.

(b) Find all functions $f(x) \in V$ that are eigenvectors of $D$, with their corresponding eigenvalues.

Solution. (a) The matrix of $D$ is
\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}.
\]

(b) The only eigenvalue of the matrix in (a) is 1, and the only eigenvectors corresponding to it are scalar multiples of $[1 \ 0 \ 0]^T$. Hence the only eigenvectors of $D$ in $V$ are scalar multiples of $e^x$, with eigenvalue 1.

9. (a) Compute the angle between the two vectors $[1 \ 0 \ 1 \ 1]^T, [0 \ 1 \ 2 \ 1]^T$ in $\mathbb{R}^4$ (with the Euclidean inner product).

(b) Find a unit vector perpendicular to both of the above vectors.

Solution. (a) We have $\cos \theta = \frac{u \cdot v}{\|u\| \cdot \|v\|} = \frac{3}{\sqrt{18}} = \frac{1}{\sqrt{2}}$. This gives $\theta = \pi/4$.

(b) The vectors perpendicular to the given ones are the nullspace of the matrix
\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 2 & 1
\end{bmatrix},
\]
which is conveniently already in row-echelon form. Solving as usual, one such vector is $[-1 \ -1 \ 0 \ 1]^T$. Divide by its norm to get a unit vector
\[
[-1/\sqrt{3} \ -1/\sqrt{3} \ 0 \ 1/\sqrt{3}]^T.
\]

10. Extend
\[
\begin{bmatrix}
1/4 \\
1/2 \\
1/4 \\
3/4 \\
1/4
\end{bmatrix},
\begin{bmatrix}
1/2 \\
0 \\
1/2 \\
-1/2 \\
1/2
\end{bmatrix}
\]
to an orthonormal basis of the space spanned by the above two vectors and
\[
\begin{bmatrix}
1 \\
2 \\
2 \\
2 \\
3
\end{bmatrix}.
\]
Solution. The required third vector is

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}
\]

(or the negative of this vector).

11. Section 4.3, Ex. 25

Solution. See textbook.