Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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Solutions to review problems for Midterm 2

1. Let V be the set of 4×4 matrices with all row- and column-sums equal to zero.

   (a) Show that V is a subspace of $M_{4,4}$.

   (b) Find $\dim(V)$.

Solution. (a) V is the solution set of a system of linear equations, hence a subspace.

(b) Method 1: Set up coordinates on $M_{4,4}$. Then we have 8 equations (one for each row and column) in 16 variables. Working out the matrix and row-reducing it shows that one equation is redundant, so the solution space has dimension $16 - 7 = 9$.

Method 2: The upper-left 3×3 block of M can be chosen at will, and six of the equations then determine the remaining entries in the first 3 rows and first 3 columns. The entry in position $(4, 4)$ is then determined by both the equation for the last row and the equation for the last column, but both give the same result. This gives a linear isomorphism from $M_{3,3}$ to V, so $\dim(V) = 9$.

2. Consider the four functions in $\mathcal{C}(\mathbb{R})$:

$$f(x) = \cos^2 x, \quad g(x) = \sin^2 x, \quad h(x) = \cos 2x, \quad j(x) = \sin 2x.$$

Are they linearly independent? Prove it if so; otherwise express one of them as a linear combination of the others.

Solution. $\cos 2x = \cos^2 x - \sin^2 x$

3. Prove that if $\mathrm{CS}(A) = \mathrm{NS}(A)$, then A is a square matrix of even size.

Solution. Let A be $m \times n$. Then $\mathrm{CS}(A)$ is subspace of $\mathbb{R}^m$, while $\mathrm{NS}(A)$ is a subspace of $\mathbb{R}^n$, so $m = n$. Now $\mathrm{rank}(A) = \dim(\mathrm{CS}(A)) = \dim(\mathrm{NS}(A)) = n - \mathrm{rank}(A)$, so $n = 2\,\mathrm{rank}(A)$, which shows that n is even.

4. Chapter 3 Review Exercise 27.

Solution. See textbook.

5. (a) Find real numbers w, x, y, z such that the characteristic polynomial of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ w & x & y & z \end{bmatrix}$$

is $(\lambda - 1)^4$.

   (b) Is the resulting matrix diagonalizable? Why or why not?

Solution. (a)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix}.$$

(b) Not diagonalizable, since its eigenvalues are all equal to 1, but it is not the identity matrix.

6. Find a 3×3 matrix X such that

$$X^2 = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}.$$

How many such matrices X are there?

Solution. Diagonalize the given matrix as $S\Lambda S^{-1}$, where

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

To find X, replace Λ by a square root, such as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \text{giving} \quad X = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

The general form of a square root of Λ is

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm 3 \end{bmatrix},$$

for a total of 8 solutions.

7. Let $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ be the Fibonacci sequence. Suppose we extend the definition of F_k to negative k by requiring that $F_k = F_{k-1} + F_{k-2}$ hold for all k.

(a) Find a matrix A such that

$$\begin{bmatrix} F_{-k} \\ F_{-(k-1)} \end{bmatrix} = A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for all k.

(b) How is F_{-k} related to F_k?

Solution. (a) $A = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. (b) $F_{-k} = (-1)^{k-1} F_k$.

8. Let $V = \operatorname{span}(\{e^x, xe^x, x^2e^x\})$.

(a) Find the matrix of the linear transformation $D\colon V \to V$ defined by differentiation, with respect to the given basis of V.

(b) Find all functions $f(x) \in V$ that are eigenvectors of D, with their corresponding eigenvalues.

Solution. (a) The matrix of D is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) The only eigenvalue of the matrix in (a) is 1, and the only eigenvectors corresponding to it are scalar multiples of $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Hence the only eigenvectors of D in V are scalar multiples of e^x, with eigenvalue 1.

9. (a) Compute the angle between the two vectors $\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 & 2 & 1 \end{bmatrix}^T$ in $\mathbb{R}^4$ (with the Euclidean inner product).

(b) Find a unit vector perpendicular to both of the above vectors.

Solution. (a) We have $\cos\theta = \mathbf{u} \cdot \mathbf{v} / \|\mathbf{u}\| \cdot \|\mathbf{v}\| = 3/\sqrt{18} = 1/\sqrt{2}$. This gives $\theta = \pi/4$.

(b) The vectors perpendicular to the given ones are the nullspace of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix},$$

which is conveniently already in row-echelon form. Solving as usual, one such vector is $\begin{bmatrix} -1 & -1 & 0 & 1 \end{bmatrix}^T$. Divide by its norm to get a unit vector

$$\begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 0 & 1/\sqrt{3} \end{bmatrix}^T.$$

10. Extend

$$\begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \\ 3/4 \\ 1/4 \end{bmatrix}, \quad \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

to an orthonormal basis of the space spanned by the above two vectors and

$$\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

Solution. The required third vector is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

(or the negative of this vector).

11. Section 4.3, Ex. 25

Solution. See textbook.