Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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Review problems for Final Exam

The final exam will be 50% on differential equations, 50% on linear algebra. Most of the review problems below are on differential equations. For the linear algebra part, I suggest you also look again at the review and exam problems from the two midterms.

Linear algebra review problems

1. Find the dimension and a basis of the space of symmetric matrices in $M_{3,3}$.

2. Show that if V and W are both 5-dimensional subspaces of $\mathbb{R}^7$, then $\dim(V \cap W) \geq 3$.

3. Let A be a real $m \times n$ matrix. Show that $\mathrm{NS}(A) \cap \mathrm{RS}(A) = \{\mathbf{0}\}$.

4. (a) Let A be an $n \times n$ matrix such that $A^2 = A$. Show that A is diagonalizable. [Hint: show that $\mathbb{R}^n$ is the sum of the $\lambda = 0$ eigenspace and the $\lambda = 1$ eigenspace of A.]

(b) Show that for a matrix A as above, we have $\mathrm{tr}(A) = \mathrm{rank}(A)$.

5. Let A be an $n \times n$ matrix with the property that the sum of the entries in every row of A is a given number c. Show that c is an eigenvalue of A and find a corresponding eigenvector. What if the columns sum to c instead?

Differential equations review problems

6. Solve the initial value problem:

$$x'(t) - t^{-1}x(t) = \sqrt{t}; \quad x(1) = 1$$

7. Solve the initial value problem:

$$x'(t) - 3x(t) = \begin{cases} 1, & \text{for } 1 \leq t < 2 \\ t - 1, & \text{for } t \geq 2; \end{cases}$$

$$x(1) = 0.$$

8. Solve the initial value problem:

$$x''(t) - 2x'(t) - 8x(t) = 6e^{4t}; \quad x(0) = 0,\ x'(0) = -5.$$

9. Solve the initial value problem:

$$x''(t) - 2x'(t) + 5x(t) = 0; \quad x(\pi/2) = -1,\ x'(\pi/2) = 1.$$

10. Solve the inital value problem:

$$x''(t) + 4x'(t) + 4x(t) = 4t^2e^{-2t}; \quad x(0) = 1,\ x'(0) = 0.$$

11. Solve the initial value problem:

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & 0 & 0 \\ 18 & 14 & 30 \\ -9 & -6 & -13 \end{bmatrix} \mathbf{x}(t); \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

12. Solve the initial value problem:

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 6 \\ -5 \end{bmatrix} e^t; \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ -4 \end{bmatrix}.$$

13. Find all solutions of the linear system

$$\mathbf{x}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}(t).$$

14. Verify that both $x_1(t) = t^3$ and $x_2(t) = |t^3|$ are solutions of the initial value problem

$$tx'(t) - 3x(t) = 0; \quad x(1) = 1,$$

which are defined, continuous and differentiable for all real numbers t. Why doesn't this contradict the existence and uniqueness theorem?

15. Show that t, t^2 and t^3 are linearly independent solutions on $(0, \infty)$ of the differential equation

$$t^3x'''(t) - 3t^2x''(t) + 6tx'(t) - 6x(t) = 0.$$

16. Find all solutions of the differential equation

$$x^{\text{vi}}(t) + 4x''(t) = 0.$$