Math H54 — Honors Linear Algebra and Differential Equations — Spring, 2004

Prof. Haiman

**Review problems for Midterm 1**

Midterm 1 is Tuesday, Feb. 24, in class, covering homework assignments 1 through 4 (Chapters 1 and 2 of the textbook). The exam will have five questions, similar in difficulty to the review problems below.

No books, notes, calculators or computers are allowed. Bring scratch paper in case you need more work space than is provided on the exam.

1. Let

$$A = \begin{bmatrix} -1 & -2 & 0 & -1 \\ 1 & 2 & 0 & 3 \\ -2 & -4 & 2 & -3 \end{bmatrix}.$$

Find a factorization $PA = LU$, where $P$ is a permutation matrix, $L$ is lower unit triangular, and $U$ is in row echelon form.

2. The matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 4 & 1 & 2 & -2 & 0 \\ -8 & -2 & -1 & 1 & 0 \\ 4 & 1 & 3 & 0 & -2 \end{bmatrix}$$

has the $PA = LU$ factorization

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using this, find all solutions of the system of linear equations

$$A\mathbf{x} = \begin{bmatrix} -1 \\ 9 \\ -21 \\ 13 \end{bmatrix}.$$

3. Let $M$ be the $3 \times 3$ matrix such that, for any $3 \times n$ matrix $A$, $MA$ is the result of adding 2 times the first row of $A$ to each of the second and third rows. Find $M$ and $M^{100}$.

4. Chapter 1, Review Exercise 16.

5. Use determinants to show that if $AB$ is invertible, then $A$ and $B$ are invertible (assuming $A$ and $B$ are square matrices of the same size).

6. (a) Prove that if $m > n$, an $m \times n$ matrix $A$ cannot have a right inverse $B$ (that is, a matrix $B$ such that $AB = I_m$). You may use the theorem we proved in the lecture that if $m < n$, then $A$ cannot have a left inverse.

(b) Give an example of an $m \times n$ matrix $A$ with $m > n$, and a left inverse of $A$.

7. If $A$ is an $n \times n$ invertible symmetric matrix, is $A^{-1}$ necessarily symmetric? Why or why not?

8. Compute the determinants of the three matrices:

$$A = \begin{bmatrix} 7 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 2 & 2 & 2 & 2 \\ -5 & 5 & 0 & 0 & 0 \\ -5 & 0 & 5 & 0 & 0 \\ -5 & 0 & 0 & 5 & 0 \\ -5 & 0 & 0 & 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 15 & 2 & 2 & 2 & 2 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

9. Find the determinant and the inverse of the matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

10. Let $X$ be an $m \times n$ matrix, and let $A$ and $B$ be the matrices given in block form as

$$A = \begin{bmatrix} I_m & X \\ X^T & -I_n \end{bmatrix}, \quad B = \begin{bmatrix} X & I_m \\ -I_n & X^T \end{bmatrix}.$$

Compute the product $AB$.

11. Chapter 2, Section 2.4, Ex. 31