Homework 7
due Friday, March 19

Reading for Lectures 14–16:
• Sections 4.1, 5.2–5.3

Problems:
• 5.2 Ex. 8, 9, 10. Use the results of these exercises to find a formula for the characteristic polynomial of $A_n$. Then compute the determinant of the $n \times n$ matrix

$$\begin{bmatrix}
2 & 1 & \ldots & 1 \\
1 & 2 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 2
\end{bmatrix}$$

• 5.2 Ex. 20, 33
• 5.3 Ex. 9, 26, 32

• Problem A. The kernel of a linear transformation $\phi: V \to W$ is defined to be the subset $\{v \in V : \phi(v) = 0\}$. The image of $\phi$ is defined to be the subset $\{\phi(v) : v \in V\} \subseteq W$. We use the notation $\ker(\phi)$ for the kernel and $\im(\phi)$ for the image.

(a) Verify that $\ker(\phi)$ is a subspace of $V$ and $\im(\phi)$ is a subspace of $W$.

(b) Assume $V$ and $W$ are finite-dimensional, and let $A$ be the matrix of $\phi$ with respect to ordered bases $B$ of $V$ and $C$ of $W$. Show that a vector $v$ belongs to $\ker(\phi)$ if and only if its coordinate vector $[v]_B$ belongs to the nullspace $\NS(A)$.

(c) With $A$ as in part (b), show that a vector $w$ belongs to $\im(\phi)$ if and only if its coordinate vector $[w]_C$ belongs to the column space $\CS(A)$.

(d) Using parts (b) and (c), show that $\dim(\im(\phi)) + \dim(\ker(\phi)) = \dim(V)$.

(e) Let $P_{<n}$ be the space of polynomials $f(x)$ of degree $< n$, and define $T: P_{<n} \to P_{<n}$ by $T(f) = f + \frac{df}{dx}$. Verify that $T$ is a linear transformation and show that $\ker(T) = \{0\}$. [Hint: verify that the nonzero solutions of the differential equation $\frac{df}{dx} + f = 0$ are not polynomials.]

(f) Using parts (d) and (e), show that every polynomial $g(x)$ can be expressed in the form $g(x) = f(x) + f'(x)$ for some polynomial $f(x)$. 

Problem B. Define $\Delta : P_{<n} \to P_{<n}$ by $\Delta f(x) = f(x + 1) - f(x)$. (This $\Delta$ is known as the forward difference operator.)

(a) Verify that $\Delta$ is a linear transformation.

(b) Describe the matrix $A$ of $\Delta$ with respect to the basis of $P_{<n}$ consisting of the polynomials $C_0, C_1, \ldots, C_{n-1}$, where $C_0(x) = 1$ and for $k > 0$,

$$C_k(x) = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

(c) For $n = 3$, find the matrix $B$ of $\Delta$ with respect to the basis $\{1, x, x^2\}$ of $P_{<3}$. Then find the change of basis matrix $X$ from the basis $\{C_0, C_1, C_2\}$ to the basis $\{1, x, x^2\}$, and verify by direct computation that $B = XAX^{-1}$. 