Math H54 — Honors Linear Algebra and Differential Equations — Spring, 2004

Prof. Haiman

Homework 7

due Friday, March 19

Reading for Lectures 14–16:

- Sections 4.1, 5.2–5.3

Problems:

- 5.2 Ex. 8,9,10. Use the results of these exercises to find a formula for the characteristic polynomial of A_n. Then compute the determinant of the $n \times n$ matrix

$$\begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 2 \end{bmatrix}.$$

- 5.2 Ex. 20, 33
- 5.3 Ex. 9, 26, 32
- Problem A. The *kernel* of a linear transformation $\phi\colon V \to W$ is defined to be the subset $\{\mathbf{v} \in V : \phi(\mathbf{v}) = \mathbf{0}\}$. The *image* of ϕ is defined to be the subset $\{\phi(\mathbf{v}) : \mathbf{v} \in V\} \subseteq W$. We use the notation $\ker(\phi)$ for the kernel and $\mathrm{im}(\phi)$ for the image.

  (a) Verify that $\ker(\phi)$ is a subspace of V and $\mathrm{im}(\phi)$ is a subspace of W.

  (b) Assume V and W are finite-dimensional, and let A be the matrix of ϕ with respect to ordered bases $\mathcal{B}$ of V and $\mathcal{C}$ of W. Show that a vector $\mathbf{v}$ belongs to $\ker(\phi)$ if and only if its coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ belongs to the nullspace $\mathrm{NS}(A)$.

  (c) With A as in part (b), show that a vector $\mathbf{w}$ belongs to $\mathrm{im}(\phi)$ if and only if its coordinate vector $[\mathbf{w}]_{\mathcal{C}}$ belongs to the column space $\mathrm{CS}(A)$.

  (d) Using parts (b) and (c), show that $\dim(\mathrm{im}(\phi)) + \dim(\ker(\phi)) = \dim(V)$.

  (e) Let $P_{<n}$ be the space of polynomials $f(x)$ of degree $< n$, and define $T\colon P_{<n} \to P_{<n}$ by $T(f) = f + \frac{df}{dx}$. Verify that T is a linear transformation and show that $\ker(T) = \{0\}$. [Hint: verify that the nonzero solutions of the differential equation $\frac{df}{dx} + f = 0$ are not polynomials.]

  (f) Using parts (d) and (e), show that every polynomial $g(x)$ can be expressed in the form $g(x) = f(x) + f'(x)$ for some polynomial $f(x)$.

- Problem B. Define $\Delta : P_{<n} \to P_{<n}$ by $\Delta f(x) = f(x+1) - f(x)$. (This Δ is known as the *forward difference operator*.)

  (a) Verify that Δ is a linear transformation.

  (b) Describe the matrix A of Δ with respect to the basis of $P_{<n}$ consisting of the polynomials $C_0, C_1, \ldots, C_{n-1}$, where $C_0(x) = 1$ and for $k > 0$,

$$C_k(x) = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

  (c) For $n = 3$, find the matrix B of Δ with respect to the basis $\{1, x, x^2\}$ of $P_{<3}$. Then find the change of basis matrix X from the basis $\{C_0, C_1, C_2\}$ to the basis $\{1, x, x^2\}$, and verify by direct computation that $B = XAX^{-1}$.