Homework 6
due Friday, March 12

Reading for Lectures 12–14:

- Sections 3.6–3.8 and 4.1

Problems:

- 3.7 Ex. 18.
- 3.8 Ex. 18. Sketch the two bases and the vector $\mathbf{x}$, and check that your answer looks right on the sketch.
- 3.8 Ex. 31.
- 4.1 Ex. 16, 28, 30, 36.
- Problem A. If W_1 and W_2 are subspaces of a vector space V, define their *sum* to be $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$. Prove that (a) the sum $W_1 + W_2$ is a subspace of V, and (b) the intersection $W_1 \cap W_2$ is a subspace of V.
- Problem B. Define matrices A and B to be *row-column equivalent* if you can get from A to B by any mixture of row and column operations.

  (a) Show that if A and B are row-column equivalent, then $\text{rank}(A) = \text{rank}(B)$.

  (b) Show that every matrix A is row-column equivalent to a matrix with the block form
  $$\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$
  where $r = \text{rank}(A)$.
- Problem C. Prove that if V is finite-dimensional and $W \subseteq V$ is a subspace, then (a) $\dim(W) \leq \dim(V)$, and (b) if $\dim(W) = \dim(V)$ then $W = V$. Part (b) of this problem gives a useful technique for showing that two vector spaces are equal if you can show that one is contained in the other.
- Problem D. (a) Show that for any $m \times n$ matrix A and $n \times l$ matrix B, the matrix equation $AB = 0$ is equivalent to the relationship $\text{CS}(B) \subseteq \text{NS}(A)$.

  (b) Let A be any matrix and let B be a matrix whose columns span the nullspace $\text{NS}(A)$, that is, $\text{CS}(B) = \text{NS}(A)$. Prove that $\text{CS}(A^T) = \text{NS}(B^T)$. Hint: use part (a) of this problem and part (b) of Problem B.

(c) Let V and W be the subspaces of $\mathbb{R}^5$ spanned by the following sets of vectors:

$$V = \text{span}\{\begin{bmatrix}1\\2\\2\\1\\3\end{bmatrix}, \begin{bmatrix}-2\\0\\1\\-3\\-2\end{bmatrix}, \begin{bmatrix}2\\2\\4\\0\\4\end{bmatrix}\}; \quad W = \text{span}\{\begin{bmatrix}6\\-2\\1\\1\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\2\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}\}.$$

Find a basis of the intersection $V \cap W$, as follows. First, use part (b) to find matrices X and Y with 5 columns each such that $V = \text{NS}(X)$ and $W = \text{NS}(Y)$ (construct them as B^T for suitably chosen matrices A). Let Z be the block matrix

$$Z = \begin{bmatrix}X\\Y\end{bmatrix}$$

and verify that $\text{NS}(Z) = \text{NS}(X) \cap \text{NS}(Y)$. Then find a basis of $\text{NS}(Z)$ as usual, using row-reduction.

Remark: As you can see, this problem is one example of a general algorithm for finding the intersection of two subspaces, when the input subspaces and the answer are specified by giving bases of them.