Math H54 | Honors Linear Algebra and Differential Equations | Spring, 2004

Prof. Haiman

Homework 3

due Friday, Feb. 13

Reminders:

- Lecture moves to 7 Evans, starting Tuesday, Feb. 10.
- No class on Thursday, Feb. 12

Reading for Lectures 6–7:

- Section 1.5 and Appendix A.1
- Sections 1.6 and 2.1–2.2

Problems:

- Section 1.5 Ex. 19, 30, 36
- 1.5 Ex. 39, 43. You can use the theorem we proved in the lecture that the product of two lower (unit) triangular matrices is lower (unit) triangular, and the corresponding theorem for upper triangular matrices.
- Section 1.6 Ex. 20, 22
- Section 2.1 Ex. 20, 21
- Section 2.2 Ex. 25, explaining your answer.
- Problem A. The discussion of existence and uniqueness of determinants in Hill, Section 2.1, page 95 is *completely wrong*. Explain why.
- Problem B. Show that if A is invertible and both A and A^{-1} have only integer entries, then $\det(A) = \pm 1$. [It follows from Cramer's rule, which we'll study later, that the converse is also true: if A has integer entries and $\det(A) = \pm 1$, then A^{-1} has integer entries.]
- Problem C. In this problem, we'll see how to factor a matrix A as $A = LPU$. This is a little different from the factorization $PA = LU$ discussed in your book.

  (a) Use lower-triangular elementary row operations, with no row switches, to reduce the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 & -1 \\ 2 & 0 & 1 & 0 & 4 \\ 4 & 1 & 3 & 7 & 4 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

to a matrix W with the property that the first column has at most one non-zero entry, and each subsequent column has at most one more non-zero entry than the column to its left. To check your work, here's the answer you should get (the problem is to find the right row operations):

$$W = \begin{bmatrix} 0 & 0 & 0 & 2 & -1 \\ 2 & 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 7 & -4 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

(b) Find a permutation matrix P and a row-echelon matrix U such that $W = PU$.

(c) Use your work in (a) and (b) to factor $A = LPU$ where L is lower unit triangular and P and U are as in (b).

One advantage of the $A = LPU$ factorization over $PA = LU$ is that it is easier to understand why it exists. In fact, it should be clear that the procedure you followed in this problem will work for any matrix A.

Another nice property is that if U has no zero rows, then the permutation matrix P in the factorization $A = LPU$ is unique. The last part of this problem shows that this is not so for the $PA = LU$ factorization.

(d) Find a matrix A that has two factorizations $P_1 A = L_1 U_1$ and $P_2 A = L_2 U_2$, with different matrices P_1 and P_2. [Hint: try any 2×2 matrix A with no zeroes in the first column.]