Math H54 Honors Linear Algebra and Differential Equations Spring, 2004

Prof. Haiman

Homework 10

due Friday, April 23

Reading for Lectures 21-23:

- Hill Section 4.2
- Boyce & DiPrima 2.1, 2.4
- For review, you might also want to read Boyce & DiPrima 2.2, 2.3

Problems:

- Boyce & DiPrima 2.1 Ex. 16, 29, 34
- Boyce & DiPrima 2.4 Ex. 3, 13, 22 (a,b), 33
- Problem A. Apply the Cauchy-Schwarz inequality for general inner product spaces to prove that for every continuous function $f\colon [-1,1] \to \mathbb{R}$, we have
$$\left(\frac{1}{2}\int_{-1}^{1} f(x)\, dx\right)^2 \leq \left(\frac{1}{2}\int_{-1}^{1} f(x)^2\, dx\right).$$
In other words, the square of the average value of $f(x)$ on $[-1,1]$ is less than or equal to the average value of $f(x)^2$.
- Problem B. Suppose we define a new inner product on the space of continuous functions $C(-1,1)$ by the formula
$$f \cdot g = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx.$$
In general, this is an improper integral, since the integrand may go to infinity at $x = \pm 1$, but it can be shown that it always converges. You may take for granted the convergence of the integral and the fact that it defines a valid inner product.

  The *Chebyshev polynomials* $T_n(x)$ are orthogonal polynomials with respect to the above inner product, where $T_n(x)$ has degree n. It is usual to normalize them so that the highest degree term of $T_n(x)$ is $2^{n-1}x^n$, except for $n = 0$, when we set $T_0 = 1$.

  (a) Compute the inner products $x^k \cdot x^l$ for $0 \leq k, l \leq 3$. To save you some trouble, here is a table of values of $I_m = \frac{1}{\pi}\int_{-1}^{1} \frac{x^m}{\sqrt{1-x^2}}\, dx$ for $m = 0, \ldots, 7$.

m	0	1	2	3	4	5	6	7
I_m	1	0	$\frac{1}{2}$	0	$\frac{3}{8}$	0	$\frac{5}{16}$	0

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  (b) Compute the Chebyshev polynomials $T_0(x)$ through $T_4(x)$.

  (c) For $m = 0, \ldots, 3$, verify the identity $T_m(\cos\theta) = \cos(m\theta)$. In fact, this identity holds for all m. Can you think of a way to prove it?