Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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Midterm 2 Solutions

1. Let $V \subseteq \mathbb{R}^3$ be the linear span of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

(a) For each vector below, either express it as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ or show that it isn't in V:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) What is the dimension of V?

Solution. (a) $\mathbf{x} = \mathbf{v}_2 - \mathbf{v}_1$. All vectors $\mathbf{v} \in V$ satisfy $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \mathbf{v} = 0$, so $\mathbf{y} \notin V$.

(b) $\dim(V) = 2$.

2. Using the basis

$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of $M_{2,2}$, find

(a) a matrix A such that $\mathrm{NS}(A)$ is the set of coordinate vectors $[X]_{\mathcal{B}}$ of symmetric matrices $X \in M_{2,2}$;

(b) a matrix B such that $\mathrm{NS}(B)$ is the set of coordinate vectors $[X]_{\mathcal{B}}$ of matrices $X \in M_{2,2}$ such that $X \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = 0$;

(c) a matrix C such that $\mathrm{NS}(C)$ is the set of coordinate vectors $[X]_{\mathcal{B}}$ of symmetric matrices $X \in M_{2,2}$ such that $X \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = 0$.

Solution. (a)

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(c)

$$C = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

3. Find a 3×3 matrix A with eigenvalues -1, 0, 1 and corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Solution. Let

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$A = S\Lambda S^{-1} = \begin{bmatrix} \frac{7}{9} & -\frac{8}{9} & \frac{4}{9} \\ -\frac{2}{9} & -\frac{8}{9} & \frac{4}{9} \\ \frac{4}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix}.$$

4. Let A be a real $n \times n$ matrix such that $\operatorname{tr}(A^2) = -1$. Is A diagonalizable using matrices with real entries? Why or why not?

Solution. The quantity $\operatorname{tr}(A^2)$ is the sum of the squares of the eigenvalues of A. Since it is negative, the eigenvalues cannot all be real.

5. If

$$\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

find a and b such that $\mathbf{w} - a\mathbf{v}_1 - b\mathbf{v}_2$ is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, which makes it easier to solve: $a = 7/10$, $b = -2/3$.

6. Let

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 4 & 2 & 0 & 2 & 4 \end{bmatrix}.$$

Find the quadratic polynomial $f(x) = ax^2 + bx + c$ that minimizes the error

$$\sum_i (y_i - f(x_i))^2.$$

Solution. We must solve the least-squares problem

$$\begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \approx \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}.$$

The answer is $f(x) = 6x^2/7 + 24/35$.