Math H54 Honors Linear Algebra and Differential Equations Spring, 2004
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First Midterm Exam Solutions

1. Given the factorization

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find all solutions of the system of linear equations

$$A\mathbf{x} = \begin{bmatrix} 14 \\ 21 \\ 34 \\ 48 \end{bmatrix}.$$

With x_3 and x_4 as free variables, the solution is

$$\mathbf{x} = \begin{bmatrix} 11 - x_3 - 3x_4 \\ -5 + 2x_4 \\ x_3 \\ x_4 \\ 4 \end{bmatrix}.$$

2. Compute the matrix product

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 6 & 6 & 6 \\ 5 & 5 & 5 \\ 4 & 4 & 4 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

This is easy if you remember that each entry of the product AB is a row of A times a column of B. The answer is

$$\begin{bmatrix} 56 & 56 & 56 \\ 56 & 56 & 56 \\ 56 & 56 & 56 \\ 56 & 56 & 56 \end{bmatrix}.$$

3. Under what conditions on a, b, c is the matrix

$$\begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

invertible?

The determinant of the given matrix is $-4a+8b-4c = -4(a-2b+c)$. Hence the matrix is invertible if and only if $a-2b+c \neq 0$.

4. Suppose that A, B, C and D are $n \times n$ invertible matrices such that $A^T BC^T = D$. Express B^T in terms of A, C and D.

$B = (A^{-1})^T D(C^{-1})^T$, so $B^T = C^{-1}D^T A^{-1}$

5. Solve the system of equations using Cramer's rule:

$$\begin{bmatrix} 2 & -3 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}.$$