Answers to review problems for 1st midterm

1. (a) \( f(n) = O(n^{2n}) \) (b) \( f(n) = O(n^6) \).

2. (a) The inner loop sets \texttt{repeat} to 1 if \( a_i \) is a repetition of an earlier element. The total gets incremented once for each non-repeated element, so at the end it counts the distinct elements.

(b) The inner loop takes \( O(n) \) steps, and the outer loop performs the inner one \( O(n) \) times, for a complexity of \( O(n^2) \).

(c) If the list is pre-sorted, we can check if \( a_i \) is a repeat by just comparing it to \( a_{i-1} \). The counting phase takes \( O(n) \) steps, so the complexity is dominated by the initial sorting phase, giving \( O(n \log n) \) overall.

3. \( \gcd(8918, 1001) = 91 \), \( \text{lcm}(8919, 1001) = 98098 \).

4. \( \gcd(2^{3\cdot 4\cdot 5 \cdot 7}, 2^7 \cdot 3^5 \cdot 11) = 2^3 \cdot 3^4 \cdot 5^3 \), \( \text{lcm}(2^{3\cdot 4\cdot 5 \cdot 7}, 2^7 \cdot 3^5 \cdot 11) = 2^7 \cdot 3^5 \cdot 5^7 \cdot 7 \cdot 11 \).

5. If \( k^2 | n \) and \( k \neq 1 \), let \( p \) be a prime factor of \( k \). Then \( p^2 | n \). Since prime factorization is unique, this contradicts the assumption that \( n \) is a product of distinct primes.

6. \( 2^{3\cdot 4\cdot 5 \cdot 7} \cdot 10 = (2 \cdot 3^2 \cdot 5^3 \cdot 7^2)^2 \).

7. Since \( 10 \equiv -4 \) and \( 15 \equiv 29 \) (mod \( 7 \)), the desired identity follows immediately.

8. \( \gcd(84, 119) = 7 = 10 \cdot 84 - 7 \cdot 119 \)

9. (a) 11 (b) \( x = 10 \).

10. (a) Passes Fermat (b) Fails Miller; hence is composite.

11. \( x \equiv 17 \) (mod \( 140 \)).

12. \( d = 343 \).

13. \( 2821 = 7 \cdot 11 \cdot 31 \). Since \( 7 - 1 = 6 \), \( 11 - 1 = 10 \) and \( 31 - 1 = 30 \) all divide 2820, Fermat’s theorem shows that \( x^{2820} \equiv 1 \) (mod \( 7 \)), (mod \( 11 \)) and (mod \( 31 \)) for all \( x \) relatively prime to 2821. By the Chinese Remainder Theorem, this implies \( x^{2820} \equiv 1 \) (mod \( 2821 \)).

14. One correct solution is \( m = 1, n = p = 2 \). A less silly solution, with the three numbers distinct and none of them equal to 1, is \( m = 2, n = 3, p = 4 \). If you demand that no two of them are relatively prime (although the problem does not require this), a solution is \( m = 6, n = 10, p = 15 \).

15. \( A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2 \). From this we see that \( A^{n+2} = -A^n \) for all \( n \), leading to the rule

\[
A^n = \begin{cases} 
A & \text{if } n \equiv 1 \pmod{4}, \\
-I_2 & \text{if } n \equiv 2 \pmod{4}, \\
-A & \text{if } n \equiv 3 \pmod{4}, \\
I_2 & \text{if } n \equiv 0 \pmod{4}.
\end{cases}
\]