# 2: If \( n \) is odd then \( n \equiv 1, 3, 5 \text{ or } 7 \pmod{8} \). In each case we have \( n^2 \equiv 1 \pmod{8} \). An alternative proof is to write \( n = 2k + 1 \), so \( n^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1 \). The number \( k(k + 1) \) is even, no matter whether \( k \) is even or odd, so 8 divides \( 4k(k + 1) \), and hence \( n^2 \equiv 1 \pmod{8} \).

# 10: Following the hint, let
\[
x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2.
\]
Then calculate
\[
x^2 + y^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2
= m^4 + 2m^2n^2 + n^4
= (m^2 + n^2)^2
= z^2.
\]
There are infinitely many choices for \( m \) and \( n \), giving infinitely many integer solutions of \( x^2 + y^2 = z^2 \).

# 10 Extra: Trying small values in the recipe for solutions above, we soon find that \( m = 1, n = 4 \) gives \((8 : 15 : 17)\). The problem doesn’t ask for a proof that this is the smallest possibility, but you could prove it by checking all possibilities with hypotenuse length less than 17.

# 25 Extra: Given consecutive odd integers \( p, p + 2, p + 4 \), observe that if \( p \equiv 0 \pmod{3} \) then \( 3 \mid p \), if \( p \equiv 1 \pmod{3} \) then \( 3 \mid p + 2 \), and if \( p \equiv 2 \pmod{3} \), then \( 3 \mid p + 4 \). In every case, one of the three numbers is divisible by 3, so they can’t all be prime unless \( p = 3 \).

# 30: We might suppose that \( p_1, \ldots, p_t \) is a list of all the primes \( \equiv 1 \pmod{4} \), and construct \( Q = 4p_1 \cdots p_t + 1 \), so \( Q \equiv 1 \pmod{4} \) and no \( p_i \) divides \( Q \). Then letting \( Q = q_1 \cdots q_t \) be the prime factorization, we might hope to show that some \( q_i \equiv 1 \pmod{4} \), yielding a contradiction. But this last conclusion doesn’t follow, since if \( t \) is even and every \( q_i \equiv 3 \pmod{4} \), we would still have \( Q \equiv 1 \pmod{4} \).