1. Chapter 8.1, #11(c). See text.

2. The graph in the previous problem is an example of an interval graph: the intersection graph of a collection of intervals in \( \mathbb{R} \). Show that an interval graph \( G \) cannot contain a 4-cycle without diagonals. In other words, if \( \{w, x, y, z\} \) are the vertices of a 4-cycle in an interval graph, then \( G \) contains at least 5 of the 6 possible edges between these vertices.

Suppose we had four intervals \( A_1 = [a_1, b_1], A_2 = [a_2, b_2], A_3 = [a_3, b_3], A_4 = [a_4, b_4] \) whose intersection graph was a 4-cycle, with edges \( \{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\}, \{A_4, A_1\} \), and no other edges. Then \( A_1 \cap A_3 = \emptyset \). After switching \( A_1 \) and \( A_3 \) if necessary (which is OK, because it’s an isomorphism from the 4-cycle to itself), we can assume that \( b_1 < a_3 \). Since \( A_2 \) overlaps both \( A_1 \) and \( A_3 \), we must have \( a_2 \leq b_1 \) and \( b_2 \geq a_3 \). Similarly, we must have \( a_4 \leq b_1 \) and \( b_4 \geq a_3 \). But then \( A_2 \) overlaps \( A_4 \), since both contain \( a_3 \). This is a contradiction.

3. Chapter 8.3, #43, #44. A possible isomorphism for #43 is

\[
\begin{align*}
 u_1 & \rightarrow v_1, \ u_2 \rightarrow v_9, \ u_3 \rightarrow v_4, \ u_4 \rightarrow v_5, \ u_5 \rightarrow v_6 \\
 u_6 & \rightarrow v_7, \ u_7 \rightarrow v_8, \ u_8 \rightarrow v_3, \ u_9 \rightarrow v_{10}, \ u_{10} \rightarrow v_2.
\end{align*}
\]

The graphs in #44 are not isomorphic. One way to prove this is to observe that the complement of the first graph consists of two disjoint 4-cycles \( (u_1u_3u_5u_7 \text{ and } u_2u_4u_6u_8) \), while the complement of the second graph is an 8-cycle \( (v_1v_4v_7v_2v_5v_8v_3v_6) \).

4. Chapter 8.6, #4, #5. See text for the path. Another path that works is \( a, c, f, i, m, p, s, z \). The length is 16.

5. In class we showed that the Ramsey number \( R(4, 4) \) is less than or equal to 18. In this exercise, we will prove that \( R(4, 4) = 18 \) by constructing a 2-coloring of the edges of \( K_{17} \) such that there is no red \( K_4 \) and no white \( K_4 \).

Our \( K_{17} \) will have vertex set \( V = \{0, 1, 2, \ldots, 16\} = \mathbb{Z}_{17} \). Color it by the rule that an edge \( \{x, y\} \) is

- red if \( y - x \in \{\pm 1, \pm 2, \pm 4, \pm 8\} \)
- white if \( y - x \in \{\pm 3, \pm 5, \pm 6, \pm 7\} \),

where all arithmetic is \((\mod 17)\).

(a) Show that the function \( f(x) = 3x \) is an isomorphism from \( K_{17} \) to itself that sends red edges to white edges and white edges to red edges. Therefore our coloring contains a red \( K_4 \) if and only if it contains a white \( K_4 \), so it is enough to prove it contains no red \( K_4 \).

Since \( 3 \cdot 6 \equiv 1 \pmod{17} \), the function \( g(y) = 6y \) is inverse to \( f(x) = 3x \). Therefore \( f \) is one-to-one and onto, and since the graph is complete, it’s an isomorphism. To show that \( f \) reverses colors, note that \( f(y) - f(x) = 3(y - x) \), so we just have to check that multiplication by \( 3 \pmod{17} \) sends the members of each set \( R = \{\pm 1, \pm 2, \pm 4, \pm 8\} \) and \( W = \{\pm 3, \pm 5, \pm 6, \pm 7\} \) to the other. This is true, since \( 3 \cdot 1 \equiv 3, 3 \cdot 2 \equiv 6, 3 \cdot 4 \equiv -5 \), and \( 3 \cdot 8 \equiv 7 \).

(b) Show that if \( w, x, y, z \) are the vertices of a red \( K_4 \), then so are \( w + a, x + a, y + a, z + a \). By taking \( a = -w \), show that if there is a red \( K_4 \) then there is one that contains vertex 0.
The color of an edge is determined by the difference of its vertices, and \((x + a) - (y + a) = x - y\), so \(\{x + a, y + a\}\) is the same color as \(\{x, y\}\) for every \(a\). Hence if \(w, x, y, z\) is a red \(K_4\), then so is \(0, x - w, y - w, z - w\).

(c) Show that if \(0, x, y, z\) are the vertices of a red \(K_4\), then so are \(0, ax, ay, az\) for any \(a \in \{\pm 1, \pm 2, \pm 4, \pm 8\}\). Also show that if \(a \in \{\pm 1, \pm 2, \pm 4, \pm 8\}\), then \(a^{-1} \in \{\pm 1, \pm 2, \pm 4, \pm 8\}\). By taking \(a = x^{-1}\), show that if there is a red \(K_4\), then there is one that contains vertices \(0\) and \(1\).

Since \(ax - ay = a(x - y)\), we see that \(\{ax, ay\}\) is the same color as \(\{x, y\}\) if multiplication by \(a\) sends the members of each set \(R\) and \(W\) into the same set. You could check this for every \(a \in R\). Or, more cleverly, observe that \(2^4 = 16 \equiv -1 \pmod{17}\), so \(R\) is the set of all powers of 2 (mod 17). From this description it is clear that if \(a\) and \(x\) are both in \(R\) then so is \(ax\). Moreover, if \(a \in R\), since multiplication by \(a\) is a bijection from \(R \cup W\) to itself, and it sends \(R\) into \(R\), it must send \(W\) into \(W\).

Similarly, you can just check that the inverse of every \(a \in R\) is in \(R\), or deduce this from the fact that \(R\) is the set of all powers of 2.

It follows that if \(0, x, y, z\) is a red \(K_4\), so \(x\) is in \(R\), then \(0, 1, x^{-1}y, x^{-1}z\) is a red \(K_4\).

(d) Find all the vertices \(x \neq 0, 1\) such that both edges \(\{0, x\}\) and \(\{1, x\}\) are red.

For \(\{0, x\}\) to be red we must have \(x \in R = \{\pm 1, \pm 2, \pm 4, \pm 8\} = \{1, 2, 4, 8, 9, 13, 15, 16\}\). For \(\{1, x\}\) to be red we must have \(x - 1 \in R\), so \(x \in \{2, 3, 5, 9, 10, 14, 16, 0\}\). The intersection of these two sets is \(\{2, 9, 16\}\), so these are the only possibilities for \(x\).

(e) Prove that there is no red \(K_4\), and therefore also no white \(K_4\), in this coloring of \(K_{17}\).

If there is a red \(K_4\) then parts (b)–(d) show that there is one of the form \(\{0, 1, x, y\}\) with \(x, y \in \{2, 9, 16\}\). But all three edges \(\{2, 9\}, \{2, 16\}\) and \(\{9, 16\}\) are white, since the differences for these edges are \(\pm 7\) and \(\pm 3\). Hence there is no red \(K_4\). By part (a), there is also no white \(K_4\).

This example shows that the Ramsey number \(R(4, 4)\) is greater than 17. In the lecture we showed that \(R(4, 4) \leq 18\), so it follows that \(R(4, 4) = 18\).