Math 55, Discrete Mathematics—Spring 2021
Final Exam

Instructions

The due deadline for this exam is Thursday, May 13 at 8:00pm PDT. This is a firm deadline! Be sure to finish your work with enough time to spare for submitting it.

Points will be deducted for late submissions. Gradescope will not accept any submissions after 11:59pm.

The exam is to be completed in a single 3 hour session at any time during the 24-hour window from the release time to the due time. The 3 hour time limit is not enforced, but please do your best to honor it. I have tried to design the exam so that this should be enough time to demonstrate your knowledge.

If you need to take a few extra minutes to finish answering a question or questions that you know how to solve, that is OK. You should not attempt to use the unenforced time limit as a way to cram in extra studying after seeing the questions. Doing so would be stressful and unlikely to significantly improve your score.

You can write answers on paper and scan in PDF format (best) or take pictures (usually lower image quality, so be sure they are readable). You can also write answers on a tablet if you have one, and save them in PDF format.

When you submit your answers, Gradescope will ask you to indicate the page or pages where your answer to each question is located. Gradescope will let you indicate one or multiple questions on each page.

This exam is open book. You may consult the textbook, your own notes, and any other books and non-interactive web resources. Calculators are allowed, but it should be possible to answer all questions doing arithmetic by hand. You may not receive assistance from another person, or give assistance, or use web resources that provide answers to questions interactively.

There are 12 questions, for a total of 100 points.

Note: This version of the exam corrects announced mistakes in the version available earlier.

UC Berkeley Honor Code

"As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others."

• If a question has a numerical answer, show enough work so that we can see how you arrived at your answer.
• Leave answers involving expressions such as factorials and binomial coefficients in unsimplified form.
• If a question asks you to prove or show something, your answer should be a logical argument written out in complete sentences. You can take as known any theorems proven in class or in the textbook.

Question 1 (8 points). Let A and B be subsets of the set of real numbers. Does $\forall a{\in}A\ \exists b{\in}B\ (a < b)$ imply $\exists b{\in}B\ \forall a{\in}A\ (a < b)$? If so, explain. If not, give a counterexample.

Question 2 (2 points each). Determine whether each of the following functions is injective (= 1-to-1), surjective (= onto), both, or neither. Briefly justify your answers.
(a) $f\colon \mathbf{Z} \to \mathbf{Z}$ given by $f(n) = n^2$ ($\mathbf{Z}$ is the set of integers),
(b) $f\colon \mathbf{N} \to \mathbf{N}$ given by $f(n) = n^2$ ($\mathbf{N}$ is the set of non-negative integers),
(c) $f\colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$ ($\mathbf{R}$ is the set of real numbers),
(d) $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ given by $f(x) = x^2$ ($\mathbf{R}_+$ is the set of positive real numbers).

Question 3 (a) (5 points). Prove that if r, s and n are positive integers, and $r^2 = s^2 n$, then n is a perfect square (that is, n is the square of an integer).
(b) (4 points). Use part (a) to prove that if n is a positive integer and not a perfect square, then $\sqrt{n}$ is irrational. Full credit on this part will be given for a correct proof assuming part (a), independent of your answer to (a).

Question 4 (8 points). Recall that the Fibonacci numbers f_n are defined by the recurrence

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \text{ for } n > 1.$$

Prove that $f_n \le 2^{n-2}$ for all $n \ge 2$.

Question 5 (a) (4 points). Find a multiplicative inverse of 47 (mod 105).
(b) (4 points). Show that 48 does not have a multiplicative inverse (mod 105).

Question 6 (8 points). Without actually doing the division, show that the number

$$999,999,999,999,999,999,999,999,999,999,999,999$$

(with 36 digits) is divisible by 37.

Question 7. Find the number of 7-letter words formed using the 26 letter alphabet A-Z, subject to each of the following restrictions. 'Word' means any sequence of 7 letters.

(a) (2 points) Words with no restrictions.

(b) (2 points) Words with all letters distinct, such as BEARFUL.

(c) (3 points) Words with the letters in alphabetical order, such as ACCRUXX.

(d) (3 points) Words with one A, 2 E's, 2 R's and 2 S's, such as ERASERS.

Question 8 (8 points). Prove the identity

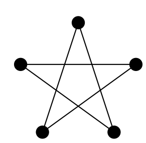
$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

for all integers $n \geq k \geq 0$. Either an algebraic or a combinatorial proof is fine.

Question 9 (8 points). One night out of four, on average, I don't get enough sleep. If I record a lecture video when I haven't gotten enough sleep, there is a 50% chance that you will understand it. If I have gotten enough sleep, there is an 80% chance that you will understand it. You watch a video and don't understand it. What is the probability that I didn't get enough sleep when I recorded it?

Question 10 (8 points). Show that if a fair 6-sided die is rolled 30 times, the probability is greater than .5 that the number of times it comes up 1 is between 3 and 7 (inclusive). Note: this question is not asking you to calculate the probability exactly.

Question 11 (8 points). Let G be a graph constructed by arranging n vertices around a circle and connecting every pair by an edge, except for pairs of vertices that are next to each other on the circle. Here is an example for $n = 5$:



For which values of n does this graph have an Euler circuit?

Question 12. Assume that the edges of a complete graph K_{17} with 17 vertices have been colored red, blue or green.

(a) (4 points). Prove that for every vertex v in the graph, there are at least six other vertices such that the edges from v to those vertices are all the same color.

(b) (5 points). Prove that, no matter how the edges have been colored, there are three vertices such that the three edges between them are all the same color.