Question 1. Mark each of the following statements True or False. No reason is required.
[Points = number correct minus three, but not less than zero.]

T (F) Every function $f : \mathbb{Z} \to \mathbb{R}$ is injective.
T (F) No function $f : \mathbb{Z} \to \mathbb{R}$ is injective.
T (F) Every function $f : \mathbb{Z} \to \mathbb{R}$ is surjective.
(T) F No function $f : \mathbb{Z} \to \mathbb{R}$ is surjective.
T (F) Every function $f : \mathbb{R} \to \mathbb{Z}$ is injective.
(T) F No function $f : \mathbb{R} \to \mathbb{Z}$ is injective.
T (F) Every function $f : \mathbb{R} \to \mathbb{Z}$ is surjective.
T (F) No function $f : \mathbb{R} \to \mathbb{Z}$ is surjective.

The two that are true are because $\mathbb{Z}$ is countable and $\mathbb{R}$ is uncountable.

Question 2. [7 pts] Prove that if A, B and C are sets such that $A \cap C \subseteq B$ and $A \subseteq B \cup C$, then $A \subseteq B$.

Let x be any element of A. We need to show that $x \in B$. There are two cases: $x \in C$, or $x \notin C$.

I) If $x \in C$, then $x \in A \cap C$, so $x \in B$ by the hypothesis $A \cap C \subseteq B$

II) If $x \notin C$, we have $x \in B \cup C$, by the hypothesis $A \subseteq B \cup C$, so $x \in B$.

In either case, $x \in B$, so the proof is complete.

Question 3. Find each of the following numbers. Express your answer in terms of binomial or multinomial coefficients, permutation numbers $P(n, k)$, and/or derangement numbers D_n, if any of these are relevant.

(a) [5 pts] The number of ways to deal hands of 7 cards to each of 5 players, from a deck of 52 cards, leaving 17 cards not dealt.

$$\binom{52}{7, 7, 7, 7, 7, 17}$$

(b) [5 pts] The number of ways to seat 6 husband-and-wife couples at a long table with six places on each side, so that the husbands are on one side, the wives are on the other, and no one sits directly across from their spouse.

$2 \cdot 6! \cdot D_6$

decide which side has the wives — seat the wives — seat the husbands, none opposite his wife.

If you assume that it has been decided in advance which side of the table is for the wives, then you would leave out the factor 2 and get the answer $6! \cdot D_6$. This answer will also be accepted as correct.

(c) [5 pts] The coefficient of x^7 in the polynomial $(2x - 3)^{10}$.

$$\binom{10}{7} 2^7 (-3)^3$$

by the binomial theorem.

Question 4. (a) [5 pts] Factor the binomial coefficient $\binom{18}{7}$ as a product of primes.

$$\frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{2 \cdot 3^2 \cdot 17 \cdot 2^4 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 2^2 \cdot 3}{7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2}$$

$$= 2^4 \cdot 3^2 \cdot 13 \cdot 17$$

(b) [5 pts] What is the largest integer m such that 6^m divides $\binom{18}{7}$?

$6^m = 2^m \cdot 3^m$ divides $\binom{18}{7}$ if and only if

$m \leq 4$ and $m \leq 2$. So the largest m is $\boxed{m=2}$.

Question 5. [7 pts] Prove that if S is a 15 element subset of $\{1, 2, \ldots, 50\}$, then there are four distinct elements $a, b, c, d \in S$ such that $a + b = c + d$. (of distinct elements)

There are $\binom{15}{2} = \frac{15 \cdot 14}{2} = 105$ pairs $\{a, b\} \subseteq S$. Each pair has sum $a+b$ in the range $1+2 = 3 \leq a+b \leq 49+50 = 99$, so there are only 97 possibilities for the sums. Hence there are two pairs $\{a, b\}$ and $\{c, d\}$ with the same sum $a+b = c+d$, by the pigeonhole principle.

If c or d was equal to a or b, then $a+b = c+d$ would imply $\{a, b\} = \{c, d\}$. Since $\{a, b\} \neq \{c, d\}$, all four elements a, b, c, d are distinct.

Question 6. [8 pts] Let f_n be the Fibonacci numbers, defined by $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Prove that

$$f_n \equiv 2n\,3^n \pmod 5.$$

We prove it by strong induction on n. For $n = 0, 1$, we have

$$f_0 = 0 \qquad 2 \cdot 0 \cdot 3^0 = 0$$

$$f_1 = 1 \qquad 2 \cdot 1 \cdot 3^1 = 6 \equiv 1 \pmod 5,$$

so the identity is true in these cases. For $n > 1$, we can assume the identity holds for $n-1$ and $n-2$. Then

$$f_n = f_{n-1} + f_{n-2} \equiv 2(n-1)3^{n-1} + 2(n-2)3^{n-2} \pmod 5.$$

We therefore need to show that

$$2n\,3^n \equiv 2(n-1)3^{n-1} + 2(n-2)3^{n-2} \pmod 5,$$

or equivalently, that

$$2n\,3^n - 2(n-1)3^{n-1} - 2(n-2)3^{n-2} \equiv 0 \pmod 5.$$

We can factor the left-hand side of this as

$$2 \cdot 3^{n-2} \cdot \left(9n - 3(n-1) - (n-2) \right)$$

$$= 2 \cdot 3^{n-2} \left(5n + 5 \right).$$

Since $5n + 5 \equiv 0 \pmod 5$ for any integer n, the desired identity is verified.

Question 7. (a) [6 pts] Find integers y and z such that

$$55y + 38z = 1.$$

Find remainders

$55 = 1 \cdot 38 + 17$

$38 = 2 \cdot 17 + 4$

$17 = 4 \cdot 4 + 1$

Back-substitute

$1 = 17 - 4 \cdot 4$

$= -4 \cdot 38 + 9 \cdot 17$

$= -13 \cdot 38 + 9 \cdot 55$

$(4 = 38 - 2 \cdot 17)$

$(17 = 55 - 38)$

i.e., $55 \cdot 9 + 38 \cdot (-13) = 1$

(b) [6 pts] Solve the congruence

$$2018\, x \equiv 10 \pmod{55}.$$

Hint: your answer to part (a) should help with part (b).

$2018 = 36 \cdot 55 + 38 \equiv 38 \pmod{55}$.

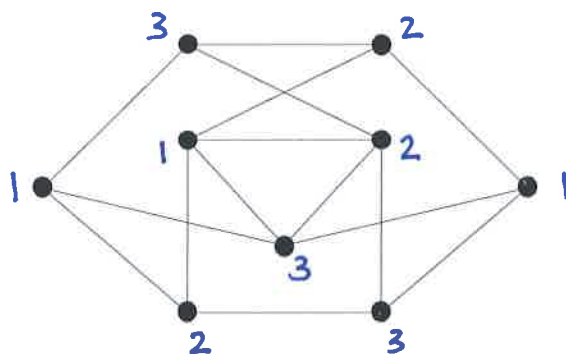By part (a), -13 is an inverse of 38 $\pmod{55}$.

To solve $38x \equiv 10 \pmod{55}$,

multiply both sides by -13 to get

$x \equiv -130$

$\equiv -20$

$\equiv 35 \pmod{55}$.

Question 8. Let G be the graph shown below.

(a) [5 pts] Show that G does not have a subgraph homeomorphic to K_5. Hint: consider the vertex degrees.

A graph homeomorphic to K_5 has 5 vertices of degree 4. If G contained such a subgraph, then G would have at least 5 vertices of degree ≥ 4. But G has only 3 vertices of degree ≥ 4.

(b) [5 pts] Prove that the graph G is not planar.

Removing the three edges that form a triangle in the middle leaves a subgraph homeomorphic to $K_{3,3}$
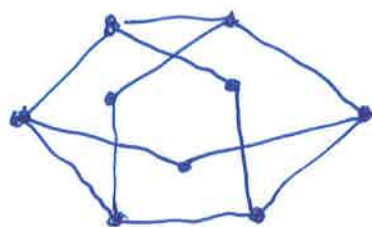
(c) [5 pts] What is the chromatic number $\chi(G)$? Justify your answer.

$\chi(G) = 3$. Since G contains a K_3, $\chi(G) \geq 3$. But G is 3-colorable, so $\chi(G) \leq 3$. See top of page for an example of a 3-coloring of G.

Question 9. [7 pts] Pavel has a bag of ten coins. Nine are fair coins, the tenth has heads on both sides. Pavel draws a coin at random from the bag and flips it. If the coin comes up heads, what is the probability that the coin Pavel drew was the two-headed coin?

Let E be the event that Pavel draws the 2-headed coin.

Let F be the event that the drawn and flipped coin comes up heads.

We want to find $p(E|F)$.

We know $p(F|E) = 1$ ← 2-headed coin must come up heads

$p(F|\bar{E}) = \frac{1}{2}$ ← other coins are fair

$p(E) = \frac{1}{10}$ ← one 2-headed coin out of 10.

By Bayes' Theorem,

$$p(E|F) = \frac{p(F|E)\, p(E)}{p(F|E)\, p(E) + p(F|\bar{E})\, p(\bar{E})}$$

$$= \frac{1 \cdot \frac{1}{10}}{1 \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{9}{10}}$$

$$= \frac{2/20}{11/20} = \frac{2}{11}$$

Question 10. (a) [3 pts] Let X be the number (from 1 to 6) that comes up when a fair six-sided die is rolled once. Find the expected value EX.

$$EX = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6$$

$$= \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$$

(We also did this in class.)

(b) [3 pts] Show that the variance $V(X)$ is equal to $35/12$.

$$E(X^2) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 9 + \frac{1}{6}\cdot 16 + \frac{1}{6}\cdot 25 + \frac{1}{6}\cdot 36$$

$$= \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

$$V(x) = E(X^2) - (EX)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}.$$

(I gave the answer so people wouldn't miss parts (c) and (d) because of an arithmetic mistake on this part.)

(c) [4 pts] Let Y be the total of the numbers that come up when a fair six-sided die is rolled 12 times. Find EY and $V(Y)$.

If X_i is the number that comes up on the ith roll,

then $Y = X_1 + X_2 + \cdots + X_{12}$, $EX_i = \frac{7}{2}$, $V(X_i) = \frac{35}{12}$.

The X_i are independent, so we can add variances; we can always add expected values. This gives

$$EY = 12\cdot\frac{7}{2} = 42 \qquad V(Y) = 12\cdot\frac{35}{12} = 35$$

(d) [4 pts] Show that the probability $P(33 \le Y \le 51)$ is at least $13/20$.

By Chebyshev's Theorem, $P(|Y-42| \ge 10) \le \frac{35}{10^2} = \frac{7}{20}$.

Then $P(33 \le Y \le 51) = P(|Y-42| < 10)$

$$= 1 - P(|Y-42| \ge 10) \ge 1 - \frac{7}{20} = \frac{13}{20}.$$