

4.1 #6 Since  $a \mid c$ , there is an integer  $k$  such that  $c = ak$ ; similarly, since  $b \mid d$ , we have  $d = bl$  for some  $l$ . Then  $cd = (ab)(kl)$  shows that  $ab \mid cd$ .

4.1 #36 Let  $a \equiv b \pmod{m}$ . Then  $a - b = km$  for some  $k \in \mathbb{Z}$ . Hence  $ac - bc = (a - b)c = kmc$ . This shows that  $m \mid ac - bc$ , i.e.  $ac \equiv bc \pmod{m}$ .

4.1 #40 Since  $n$  is odd, let  $n = 2k+1$ . Then  $n^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$ . Now, one of  $k$  and  $k+1$  is even, so  $k(k+1)$  is even, i.e.  $2 \mid k(k+1)$ , and therefore (by Problem 6, above, in fact)  $8 \mid 4k(k+1)$ , i.e.  $8 \mid n^2 - 1$ . This shows  $n^2 \equiv 1 \pmod{8}$ . [Another way to solve the problem is to observe that if  $n$  is odd, then  $n \pmod{8} \in \{1, 3, 5, 7\}$ , and then calculate  $1^2 = 3^2 = 5^2 = 7^2 = 1$  in  $\mathbb{Z}_8$ .]

$$4.2 \#28 \quad 123^{1001} \pmod{101} = 22$$

4.3 #32 Let  $n = d_k d_{k-1} \dots d_1 d_0$  in base 10, i.e.

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \dots + d_1 \cdot 10 + d_0 \cdot 10^0.$$

Now  $10 \equiv -1 \pmod{11}$ , so  $10^m \equiv (-1)^m \pmod{11}$ , i.e.

$10^m \equiv 1 \pmod{11}$  if  $m$  is even,  $10^m \equiv -1 \pmod{11}$  if  $m$  is odd.

$$\begin{aligned} \text{Hence } n &\equiv (-1)^k d_k + (-1)^{k-1} d_{k-1} + \dots + d_2 - d_1 + d_0 \pmod{11} \\ &\equiv (d_0 + d_2 + d_4 + \dots) - (d_1 + d_3 + d_5 + \dots). \end{aligned}$$

$$\begin{aligned} \text{In particular, } 11 \mid n &\iff n \equiv 0 \pmod{11} \iff \\ (d_0 + d_2 + \dots) - (d_1 + d_3 + \dots) &\equiv 0 \pmod{11} \iff \end{aligned}$$

$$11 \mid ((d_0 + d_2 + \dots) - (d_1 + d_3 + \dots))$$