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2.1 #30 $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

2.2 #14 Using $A = (A - B) \cup (A \cap B)$ and $B = (B - A) \cup (A \cap B)$,

we find $A = \{1, 5, 7, 8, 3, 6, 9\}$, $B = \{2, 10, 3, 6, 9\}$. (And then we can see that $A - B$, $B - A$ and $A \cap B$ are as specified in the problem—which in principle we should check, rather than trust the author to have given us an error-free problem!)

2.2 #24 To prove $(A - B) - C = (A - C) - (B - C)$ I will show that each set is a subset of the other.

~~X, Y, Z are elements of A-B, we have X-Z > Y-Z > Z follows easily from the definition of the difference of sets. Applying this~~

To show $(A - B) - C \subseteq (A - C) - (B - C)$, let $x \in (A - B) - C$.

Then $x \in A - B$, so $x \in A$, and $x \notin C$, so $x \in A - C$. Since $(A - B) - C \subseteq A - B$ (by the definition of set difference), $x \notin B$, and therefore $x \notin B - C$, since $B - C \subseteq B$. Having shown that $x \in A - C$ and $x \notin B - C$, we have shown that $x \in (A - C) - (B - C)$.

To show $(A - C) - (B - C) \subseteq (A - B) - C$, let $x \in (A - C) - (B - C)$.

Then $x \in A$, since $(A - C) - (B - C) \subseteq A - C \subseteq A$, and $x \notin C$ since $(A - C) - (B - C) \subseteq A - C$. If x were a member of B it would therefore be a member of $B - C$, contrary to the definition of $(A - C) - (B - C)$. So $x \notin B$, and therefore $x \in A - B$. Since we also know $x \notin C$, we have $x \in (A - B) - C$.

Several other ways to solve this problem are: using a Venn diagram or membership table; defining predicates $a(x)$, $b(x)$, $c(x)$ to mean ~~the~~ $x \in A$, $x \in B$, $x \in C$ and

showing $(a \wedge \neg b) \wedge \neg c$ is logically equivalent to
 $(a \wedge c) \wedge \neg (b \wedge \neg c)$; or showing that $(x \neg y) \neg z = x \neg (y \vee z)$
and using this to show that both $(A \neg B) \neg C$ and $(A \neg C) \neg (B \neg C)$
are equal to $A \neg (B \vee C)$.

2.3 #20 a) $f(x) = 2x$

b) $f(x) = \begin{cases} 0 & \text{if } x=0 \\ x-1 & \text{if } x>0 \end{cases}$

c) $f(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x=1 \\ x & \text{if } x>1 \end{cases}$

d) $f(x)=0$ for all $x \in \mathbb{N}$.

{Many other solutions are possible, of course.]

2.3 #40

(a) To show $f(S \cup T) = f(S) \cup f(T)$, we'll show each set contains the other. Clearly $f(S)$ and $f(T)$ are subsets of $f(S \cup T)$, since if $b = f(a)$ with $a \in S$, then $a \in S \cup T$ as well, and similarly if $a \in T$. Hence $f(S) \cup f(T) \subseteq f(S \cup T)$. To show $f(S \cup T) \subseteq f(S) \cup f(T)$, let $b \in f(S \cup T)$. Then $b = f(a)$ for some $a \in S \cup T$. If $a \in S$, then $b \in f(S)$. If $a \in T$, then $b \in f(T)$. Hence $b \in f(S) \cup f(T)$.

(b) To show $f(S \cap T) \subseteq f(S) \cap f(T)$, let $b \in f(S \cap T)$. Then $b = f(a)$ for some $a \in S \cap T$. Since $a \in S$ and $a \in T$, we have $b \in f(S)$ and $b \in f(T)$, i.e. $b \in f(S) \cap f(T)$.

Note that the reverse inclusion need not hold, for example if $f : \mathbb{N} \rightarrow \mathbb{N}$ is the constant function $f(x)=0$ and $S = \{0\}$, $T = \{1\}$, then $f(S \cap T) = f(\emptyset) = \emptyset$, but $f(S) \cap f(T) = \{0\} \cap \{0\} = \{0\}$.

2.5 #2 (c) $\{n \in \mathbb{Z} \mid |n| \leq 1,000,000\}$ is finite

(a) $(0, 2) \subseteq \mathbb{R}$ is uncountable

(e) $\{2, 3\} \times \mathbb{Z}_{\geq 0}$ is countable. We can enumerate its elements in sequence as

$(2, 1), (3, 1), (2, 2), (3, 2), (2, 3), (3, 3), \dots$

In symbols, a one-to-one correspondence (bijective function)

$$f: \mathbb{N} \rightarrow \{2, 3\} \times \mathbb{Z}_{\geq 0}$$

is given by

$$f(n) = \begin{cases} (2, \frac{n}{2} + 1) & n \text{ even} \\ (3, \frac{n+1}{2}) & n \text{ odd} \end{cases}.$$

2.5 #18 If $|A|=|B|$, it means there exists a bijective function $f: A \rightarrow B$.

Define $F: P(A) \rightarrow P(B)$ by

$F(S) = f(S)$ for each $S \in P(A)$, i.e. each $S \subseteq A$. I claim that if $g: B \rightarrow A$ is the inverse of f , then $G: P(B) \rightarrow P(A)$ defined by $G(T) = g(f^{-1}(T))$ is inverse to F , hence F is bijective.

To see this, note that $g(f(S)) \stackrel{\text{def}}{=} \{g(b) : b \in f(S)\} = \{g(f(a)) : a \in S\} = (g \circ f)(S)$. But $g \circ f = \text{id}_A$,

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by defn of $f(S)$

so $(g \circ f)(S) = S$. This shows $G(F(S)) = (g \circ f)(S) = S$ for all $S \in P(A)$, i.e. $G \circ F = \text{id}_{P(A)}$. Switching the

roles of f and g , the same reasoning shows $F \circ G = \text{id}_{P(B)}$.