

Math 55 Fall 2012
HW 2 Solutions

1.8 #4 Theorem For all real numbers a, b, c , we have

$$\min(a, \min(b, c)) = \min(\min(a, b), c).$$

Proof. We'll show that both sides are equal to the smallest of the three numbers, i.e., to $\min(a, b, c)$, by considering the three possibilities for which of the three numbers is smallest.

Case 1: $\min(a, b, c) = a$, i.e. $a \leq b$ and $a \leq c$.

Then $\min(b, c)$ is either b or c , so $a \leq \min(b, c)$, and therefore $\min(a, \min(b, c)) = a$. On the other side, $\min(a, b) = a$, so $\min(\min(a, b), c) = \min(a, c) = a$.

So $\min(a, \min(b, c)) = a = \min(\min(a, b), c)$.

Case 2: $\min(a, b, c) = b$, i.e. $b \leq a$ and $b \leq c$.

Then $\min(b, c) = b$, $\min(a, \min(b, c)) = \min(a, b) = b$, and $\min(a, b) = b$, $\min(\min(a, b), c) = \min(b, c) = b$.

So $\min(a, \min(b, c)) = b = \min(\min(a, b), c)$.

Case 3₁ is the same as Case 1 with a and c switched.

(i.e., $\min(a, b, c) = c$)

□

Note that the cases are not mutually exclusive: the numbers a, b, c do not have to be distinct, so two of them could be equal to the minimum. This is OK because the cases cover all possibilities: at least one of a, b, c must be equal to $\min(a, b, c)$.

1.8 #20 Theorem Given any real x , there exists a unique integer n and real number ε such that $0 \leq \varepsilon < 1$ and $x = n + \varepsilon$.

Proof. Existence: let n be the largest integer less than or equal to x . Let $\varepsilon = x - n$. Then $x = n + \varepsilon$ holds, and $\varepsilon \geq 0$ because $n \leq x$. It remains to prove that $\varepsilon < 1$. Now, $n+1$ is an integer larger than n , so we cannot have $n+1 \leq x$, since we assumed n was the largest such integer. Therefore, $n+1 > x$, i.e. $n-x < 1$, so $\varepsilon < 1$.

Uniqueness: Suppose $x = n + \varepsilon$ and also $x = m + \delta$, where m, n are integers, and $0 \leq \varepsilon, \delta < 1$. Then $m-n = (x-\delta) - (x-\varepsilon) = \varepsilon - \delta$. Since ε and δ are both in the interval $[0, 1)$, $|\varepsilon - \delta| < 1$, hence $|m-n| < 1$. But $m-n$ is an integer, so $|m-n| < 1$ implies $m-n=0$, i.e. $m=n$. Then we also have $\varepsilon = \delta$, since $\varepsilon = x-n$ and $\delta = x-m$. This shows that the two solutions (n, ε) and (m, δ) are the same, i.e. the solution is unique. \square

1.8 #26 First note that we can never reach nine ones: for that to happen, at the previous step adjacent bits must be unequal, i.e., they alternate 0101... around the circle. But since the number of positions (nine) is odd, that is not possible.

Now suppose we reach nine zeroes. At the previous step we must have all bits equal, i.e. nine zeroes or nine ones. But we already ruled out nine ones, so the only way to reach nine zeroes is if we already had nine zeroes at the previous step. Since the starting position is not nine zeroes or nine ones, it follows that the position at every step is not nine zeroes.

1.8 #30. Since $(-x)^2 = x^2$ and $(-y)^2 = y^2$, we can assume WLOG that if there is a solution, then there is one with $x, y \geq 0$. Since $5 \cdot 2^2 > 14$, the only possibilities for y are 0 and 1. If $y=0$, then x must satisfy $2x^2 = 14$, i.e. $x^2 = 7$. This is not possible since 7 is not a square. If $y=1$, then x must satisfy $2x^2 = 9$. This is not possible since 9 is odd.

1.8 #34 Theorem $\sqrt[3]{2}$ is irrational.

Proof. [The idea is to adapt the proof we know that $\sqrt{2}$ is irrational.]

Suppose $\sqrt[3]{2}$ were rational. Then there must exist integers p, q such that $\sqrt[3]{2} = p/q$. Writing this fraction in lowest terms, we can assume p and q have no common factor.

$$\text{Cubing, we get } 2 = p^3/q^3$$

$$p^3 = 2q^3.$$

Then p^3 is even. But the product of odd numbers is odd, so the cube of an odd number is odd, so p must be even, say $p = 2k$. Then we have

$$(2k)^3 = 8k^3 = 2q^3$$

$$4k^3 = q^3.$$

This implies q^3 is even, hence (by the same reasoning as above) q is even. But then p and q have 2 as a common factor, a contradiction. □

Note that one can replace cubes by n^{th} powers and prove by essentially the same argument not only that $\sqrt{2}$ and $\sqrt[3]{2}$ are irrational, but that $\sqrt[n]{2}$ is irrational for every integer $n \geq 1$.

2.1 #18 Let $A = \emptyset$, $B = \{\emptyset\}$. This is the simplest example, but many others are possible.

2.1 #22 Yes, if $P(A) = P(B)$ then $A = B$. To prove this, note that $P(A) = P(B)$ means A and B have the same subsets. Now $A \subseteq A$, so it follows that $A \subseteq B$. Similarly, since $B \subseteq B$, it follows that $B \subseteq A$. But $A \subseteq B$ and $B \subseteq A$ imply that every member of A is a member of B and conversely. Thus A and B have the same members, so $A = B$.

2.1 #44(c) The truth set $\{x \in \mathbb{Z} \mid x < x^2\}$ consists of all integers except 0 and 1. Indeed, $0 = 0^2$ and $1 = 1^2$, so the predicate $R(x) : x < x^2$ has $R(0)$ false and $R(1)$ false. On the other hand, if $x > 1$, then multiplying the inequality $x > 1$ by the (positive) number x gives $x^2 > x$, while if $x < 0$, then $x < 0 < x^2$, since x^2 is positive. So for every $x \in \mathbb{Z}$ other than 0 or 1, $R(x)$ is true.

Additional Problem:

Theorem For all real x, y, z , at least one of xy , xz and yz is ≥ 0 .

Proof. We'll consider cases :

- If x and y are both ≥ 0 or both ≤ 0 , then $xy \geq 0$.

- Otherwise, one of x and y is ≥ 0 and one is ≤ 0 .

WLOG $x \geq 0$ and $y \leq 0$.

- Then, if $z \geq 0$, $xz \geq 0$

- Otherwise, if $z \leq 0$, $yz \geq 0$.

□