

$\frac{}{\therefore}$

1.2 #30 : A knave cannot say "I am not the spy," since it would be true. So C must be the knave, as neither A nor B can be. Then C's statement "A is the spy" is false. Hence B is the spy, and A is the knight.

1.3 #24 To show $(p \rightarrow q) \vee (p \rightarrow r) \Leftrightarrow p \rightarrow (q \vee r)$, we may either use a truth table (with 8 rows), or reason as follows: $p \rightarrow (q \vee r)$ is false if and only if p is true and $q \vee r$ is false (by def. of " \rightarrow "). In turn, $q \vee r$ is false if and only if q and r are both false (def. of " \vee "). So $p \rightarrow (q \vee r)$ is false if p is true, q is false and r is false; otherwise $p \rightarrow (q \vee r)$ is true.

For comparison, $(p \rightarrow q) \vee (p \rightarrow r)$ is false if and only if $p \rightarrow q$ and $p \rightarrow r$ are both false (def. of " \vee "), and thus if and only if p is true and both q and r are false (def. of " \rightarrow ").

Hence $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.

1.4 #44 $\forall x (P(x) \leftrightarrow Q(x))$ and $\forall x P(x) \leftrightarrow \forall x Q(x)$ are not logically equivalent. Here is an example in which $\forall x (P(x) \leftrightarrow Q(x))$ is false, but $\forall x P(x) \leftrightarrow \forall x Q(x)$ is true:

let the domain be $\{1, 2\}$, let $P(x)$ be " $x=1$ ", and let $Q(x)$ be " $x=2$ ". Then $\forall x P(x)$ and $\forall x Q(x)$ are both false, so $\forall x P(x) \leftrightarrow \forall x Q(x)$ is true. But $P(x) \leftrightarrow Q(x)$ is false for at least one x in the domain (actually, for both values of x , in this example), so $\forall x (P(x) \leftrightarrow Q(x))$ is false.

1.5 #28 a) True (every x in \mathbb{R} has a square)

b) False (not every x in \mathbb{R} has a square root)

c) True (every non-zero x in \mathbb{R} has a multiplicative inverse $y = 1/x$).

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these

d) False (addition is commutative in \mathbb{R} , but the proposition says it isn't)

e) True ($x=0$ has the property $\forall y (xy=0)$).

1.5 #44 $\forall a \forall b \forall c \forall x \forall y \forall z$

$((\neg(a=0 \wedge b=0 \wedge c=0) \wedge$

$ax^2 + bx + c = 0 \wedge$

$ay^2 + by + c = 0 \wedge$

$az^2 + bz + c = 0) \rightarrow (x=y \vee x=z \vee y=z)$).

[domain of all quantifiers taken to be \mathbb{R}]

In words, we are saying that given the coefficients a, b, c of a quadratic polynomial f , such that the polynomial f is not identically zero (this is the $\neg(a=0 \wedge b=0 \wedge c=0)$ part), and three roots x, y , and z of f , some two of the three roots are equal (this the $(x=y \vee x=z \vee y=z)$ part).

1.7 #8 I'll use an indirect proof: supposing both n and $n+2$ are squares, we'll reach a contradiction.

So, let $n = k^2$, $n+2 = l^2$.

Then $l^2 - k^2 = n+2 - n = 2$.

But, also, $l^2 - k^2 = (l-k)(l+k)$.

We can assume without loss of generality that k and l are ≥ 0 , since every square is the square of some non-negative integer. Then, since $n+2 > n$, we know that $l > k$. So $2 = (l-k)(l+k)$ is a factorization of 2 into positive integer factors. Hence we either have $l-k=1$, $l+k=2$, or $l-k=2$, $l+k=1$. But this is impossible, since neither of these has a solution in integers (the unique solution of $l-k=1$, $l+k=2$ is $l=\frac{3}{2}$, $k=\frac{1}{2}$, while the unique solution of $l-k=2$, $l+k=1$ is $l=\frac{3}{2}$, $k=-\frac{1}{2}$).

1.7 #32 I'll take as known the fact that the sum and product of rational numbers is rational. The sum was done in Example 7, and the product can be done in a similar way. Now we'll prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: Assume x is rational. Then $x/2 = x \cdot \frac{1}{2}$ is a product of rational numbers, hence rational.

$(ii) \Rightarrow (iii)$: Assume $\frac{x}{2}$ is rational. Then $3x = 6 \cdot \frac{x}{2}$ is rational (product of rationals), hence $3x-1 = 3x + (-1)$ is rational (sum of rationals).

(iii) \Rightarrow (i) Assume $3x - 1$ is rational. Then $3x = (3x - 1) + 1$ is rational (sum), hence $x = \frac{1}{3} \cdot 3x$ is rational (product).

1.7 #34 The reasoning must be wrong because it concludes that $x=1$ and $x=-1$ are both solutions to $\sqrt{2x^2-1} = x$, but in fact $x=-1$ is not a solution (by convention, the sign $\sqrt{}$ is always understood to stand for the non-negative square root of a real number; otherwise it would be ambiguous).

The error is that step (2) is not reversible : from $\sqrt{2x^2-1} = x$ we can correctly deduce $2x^2-1 = x^2$, but in the reverse direction, from $2x^2-1 = x^2$ we can only deduce $\sqrt{2x^2-1} = \pm x$. Thus the reasoning shows that no solution other than $x=\pm 1$ exists, but it does not show that $x=\pm 1$ are both solutions.

The reasoning given in the problem, plus the verification that $x=1$ is a solution and $x=-1$ is not, correctly shows that $x=1$ is the unique solution.