Math 55: Discrete Mathematics, Fall 2012
Final Exam Solutions

1. (8 pts) Solve the simultaneous congruences

$$x \equiv 3 \pmod 8$$
$$x \equiv 5 \pmod 9$$

Solution:

$$x \equiv 59 \pmod{72}$$

2. (4 pts each) For each of the following give a yes or no answer and a one-sentence justification.

(a) Does an algorithm A exist which takes as input a program P and additional data I, and halts if and only if P halts when run with input I?

Yes. The algorithm can trace the steps of P on I and halt if P does.

(b) Does an algorithm B exist which takes as input a program P and additional data I, and halts if and only if P *does not* halt when run with input I?

No. If this were possible we could solve the halting problem for P on I by running A and B together and seeing which one halts.

3. (5 right 8 pts, 4 right 5 pts, 3 right 2 pts) Let S be the set of all finite subsets of $\mathbb{Z}$. Mark each of the following True or False. Here $\mathbb{Z}$ denotes the set of all integers, and $\mathcal{P}(A)$ denotes the power set of A.

(a) $\mathbb{Z} \in S$ False

(b) $\mathbb{Z} \subseteq S$ False

(c) $S \subseteq \mathcal{P}(\mathbb{Z})$ True

(d) $\mathcal{P}(\mathbb{Z}) \subseteq S$ False

(e) $S \cap \mathbb{Z} = \emptyset$ True

4. (8 pts) Do there exist irrational numbers x and y such that $x + y$ is rational? Prove that your answer is correct.

Yes. To prove it we need only give an example, for instance $x = \sqrt{2}$, $y = -\sqrt{2}$, $x + y = 0$.

5. (8 pts) Which of the following is the most reasonable estimate for the number of steps Pollard's algorithm might take to factor the number $n = 72916997$: 10 steps, 100 steps, 1000 steps, 10,000 steps, 100,000 steps, or 1,000,000 steps?

Justify your choice. You may assume that the given number n is a product of two 4-digit primes.

The typical number of steps for Pollard is approximately the square root of the smallest factor. The two 4-digit factors of 72916997 must be between 7292 and 9999, so the best estimate among the given choices for the number of steps is 100.

6. (5 pts each) Find the number of ways to assign 75 students to 3 discussion sections
(a) if each section must have 25 students;

$$\binom{75}{25,\ 25,\ 25} = \frac{75!}{(25!)^3}$$

(b) if there can be any number of students (including zero) in each section.

$$3^{75}$$

7. (8 pts) What is the largest number of elements that a set of integers can have if it does not contain three distinct elements a, b, c such that $a \equiv b \equiv c \pmod{10}$?

$$20$$

8. (8 pts) Evaluate

$$\binom{7}{0} - 2\binom{7}{1} + 2^2\binom{7}{2} - 2^3\binom{7}{3} + 2^4\binom{7}{4} - 2^5\binom{7}{5} + 2^6\binom{7}{6} - 2^7\binom{7}{7}.$$

By the binomial theorem, this expression is equal to $(1-2)^7 = (-1)^7 = -1$.

9. (8 pts) January 1, 2011 was a Saturday. Find the probability that a baby born in 2011 was born on Sunday, on Monday, and so on for each of the seven days of the week. Assume that the baby's birthday is equally likely to be any of the 365 days in the year.

Each of the seven weekdays occured 52 times in the last $364 = 52 \cdot 7$ days of 2011. January 1 makes one extra Saturday. Therefore, the probability of being born on Saturday is 53/365; on each of the other six days it is 52/365.

10. (5 pts each) Let X be the number that comes up when a fair 6-sided die is rolled once. We calculated $EX = 7/2$ and $V(X) = 35/12$ in class.
(a) Let Y be the total when the die is rolled 12 times. Find EY and $V(Y)$.

$$EY = 12\,EY = 42, \quad V(Y) = 12\,V(X) = 35.$$

(b) Use the values of EY and $V(Y)$ to find a lower bound on the probability that $33 \le Y \le 51$.

The complementary event is $|Y - EY| \ge 10$. By Chebyshev's Theorem,

$$P(33 \le Y \le 51) = 1 - P(|Y - EY| \ge 10) \ge 1 - V(Y)/10^2 = .65$$

11. (8 pts) Find the probability that a number chosen at random between 1 and 100 is divisible by 5 or 7.

Of the numbers $1 \le n \le 100$, 20 are divisible by 5, $\lfloor 100/7 \rfloor = 14$ are divisible by 7, and $\lfloor 100/35 \rfloor = 2$ are divisible by both. Hence $20 + 14 - 2 = 32$ are divisible by 5 or 7, giving a probability of .32.

12. (8 pts) There are 5 waiters in a restaurant. One of them is color-blind. If you order tomato soup from the color-blind waiter there is a 50 percent chance that he will bring pea soup instead. You order tomato soup and receive the right soup. What is the probability that your waiter is the color-blind one?

Assume that the waiter for your table was chosen at random, and that only the color-blind waiter might bring the wrong soup.

Let E be the event that you get the right soup and F the event that you have the color-blind waiter. We are given the information that $P(F) = 1/5$, $P(E|F) = 1/2$ and $P(E|\overline{F}) = 1$. We want to find $P(F|E)$. By Bayes' Theorem,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\overline{F})P(\overline{F})} = \frac{(1/2)(1/5)}{(1/2)(1/5) + 1(4/5)} = \frac{1}{9}.$$