Math 55: Discrete Mathematics, Fall 2008
Homework 13 Solutions

* 8.5: 16. Reflexive: if $(c, d) = (a, b)$, the relation becomes $ab = ba$, which is true. Symmetric: the relation $ad = bc$ is unchanged if we switch (a, b) with (c, d). Transitive: suppose $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then $ad = bc$ and $cf = de$. Multiplying the first equation by f and the second by b, we get $adf = bcf = bde$. Since the letters stand for positive integers, we can cancel d to get $af = be$, thus $(a, b)\ R\ (e, f)$.

24 (a) Not an equivalence relation, because not symmetric.

(b) Equivalence relation.

* [5 pts each part] 40 (a) $[(1, 2)]$ is the set of pairs $(c, 2c)$ for positive integers c.

(b) For each positive rational number r, there is an equivalence class consisting of all pairs (a, b) of positive integers such that $b/a = r$. Every class is of this form.

(This exercise hints at a deeper idea: the actual *definition* of the rational number system represents a rational number as an equivalence class of fractions b/a, where b and a are integers with $a \neq 0$.)

46 (a) Partition.

(b) Partition.

(c) Not a partition, because the closed intervals overlap at their endpoints.

(d) Not a partition, because the union is not all of $\mathbb{R}$. The integers are omitted.

(e) Partition.

(f) Partition.

54. P_1 is a refinement of P_2 means that each block of P_1 is contained in a block of P_2. Since the blocks of P_i are the equivalence classes of R_i, this holds if and only if $[x]_1 \subseteq [x]_2$ for all $x \in A$, where $[x]_i$ denotes the equivalence class of x with respect to R_i. In turn, the latter condition holds if and only if $x\ R_1\ y$ implies $x\ R_2\ y$ for all $x, y \in A$, that is, if and only if $R_1 \subseteq R_2$.

(A) Let S^* be the transitive closure of the reflexive and symmetric closure $S = R \cup R^{-1} \cup \Delta$. Then S^* is reflexive by 8.4, Exercise 22, symmetric by 8.4, Exercise 23, and transitive by definition, so S^* is an equivalence relation.

We must now show that S^* is contained in any equivalence relation T that contains R. Since T is reflexive and symmetric and contains R, T contains S. Then, since T is transitive, T contains S^*.

(B) An example of a relation R on the set $\{1, 2, 3\}$ such that the reflexive and symmetric closure of the transitive closure R^* is not an equivalence relation is the relation $R = \{(1, 2), (1, 3)\}$. The relation R is transitive to begin with, so $R = R^*$. The reflexive and symmetric closure of R^* is then $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$, which is not transitive.

* (C) To prove that

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$

for $k, n > 0$, let X be the set of partitions of $\{1, \ldots, n\}$ into k blocks, so $S(n,k) = |X|$. Let $X_1 \subseteq X$ be the subset consisting of those partitions in which $\{n\}$ is a block, and let $X_2 \subseteq X$ be the complementary subset consisting of partitions in which the block containing n has more than one element. A partition P in X_1 is determined by its restriction to $\{1, \ldots, n-1\}$, which is an arbitrary partition with $k-1$ blocks. Therefore $|X_1| = S(n-1, k-1)$. The restriction to $\{1, \ldots, n-1\}$ of a partition Q in X_2 is an aribitrary partition Q' with k blocks. To determine Q we must choose which block of Q' to add n to, in k ways. Therefore $|X_2| = kS(n-1, k)$. The formula now follows, since $|X| = |X_1| + |X_2|$.

The table constructed using the relation above is

	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	1	2	1	0	0	0	0
4	0	1	7	6	1	0	0	0
5	0	1	15	25	10	1	0	0
6	0	1	31	90	65	15	1	0
7	0	1	63	301	350	140	21	1

.

By comparison, the formula gives $S(7,3) = (3^7 - \binom{3}{1}2^7 + \binom{3}{2})/6 = 301$.